ON RIESZ SUMMABILITY

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ON RIESZ SUMMABILITY

being a Thesis presented by

Bruce L.R. Shawyer, B.Sc.

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in application for the degree of

Doctor of Philosophy.

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DECLARATION

I hereby declare that the following Thesis is based on results obtained by me, that the Thesis is my own composition and that it has not previously been presented in application for a Higher Degree.

The research was carried out in St. Salvator's College, in the University of St. Andrews.

Signed

CERTIFICATE

I certify that Bruce L.R.Shawyer has spent nine terms at research work in St.Salvator's College in the University of St.Andrews, that he has fulfilled the conditions of Ordinance No. 16(St.Andrews) and that he is qualified to submit the accompanying Thesis in application for the degree of Doctor of Philosophy.

Signed

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(Supervisor)

CAREER

I matriculated in St.Salvator's College in the University of St. Andrews in October, 1955, and followed a course leading to graduation in Science until June, 1960 .

In July, 1960, I commenced the research on Riesz summability, which is now being submitted in this Thesis in application for the degree of Doctor of Philosophy.

I was appointed to a D.S.I.R. research studentship.

Contents.

Chapter I.

Introduction.

Preface.

The thesis is divided into four chapters. The first contains notation and fundamental results. The others contain a number of theorems on Riesz summability, ordinary in the second, absolute in the third and strong in the fourth.

The substance of chapter II has appeared in the Proceedings of the Glasgow Mathematical Association [2].

I wish to express my sincere thanks to my supervisor, Dr. D. Borwein, for suggesting the problems and for his most valuable advice and criticism. Also, I wish to express my sincere thanks to the Department of Scientific and Industrial Research for awarding me a research studentship, and so, enabling me to perform the work for this thesis.

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[2] D.Borwein and B.L.R.Shawyer, "On Riesz summability factors", Proc. Glasgow Math. Assoc., 5, (1961 - 62), 188 - 196.

1.1 Introduction.

The notation introduced in this chapter is defined for the whole thesis. Other notation will be introduced as required in each subsequent chapter.

In all cases, $\int_a^b f \, d\phi$ is taken to be a Riemann-Stieltjes integral, since we only consider cases in which f is continuous and ϕ is of bounded variation.

All products, $\prod_{\nu=1}^{r} g_\nu$, are defined to have the value one, whenever $r = 0$.

1.2 Notation, definitions and basic results.

Suppose that a, k are positive numbers, and that p is the integer such that

$$k-1 \leq p < k .$$

Let M, M_1 be finite constants, not necessarily the same at each occurence, and let M be positive.

Let $\phi(w)$, $\psi(w)$ be functions defined in $[0,\infty)$ with absolutely continuous $(p+1)^{th}$ derivatives in every interval $[0,W]$, and let $\phi(w)$ be unboundedly increasing.

For a given infinite series $\sum_{n=1}^{\infty} a_n$ and an unboundedly increasing sequence of positive numbers $\lambda = \{\lambda_n\}$, Riesz sums are defined by

$$A_k(w) = \begin{cases} \sum_{\lambda_n < w} (w - \lambda_n)^k a_n & \text{if } \lambda_1 < w , \\ 0 & \text{otherwise} , \end{cases}$$

and we write

$$A(w) = A_0(w) .$$

The Riesz mean of type λ and order k is defined to be

$$C_k(w) = w^{-k} A_k(w) .$$

If

$$w^{-k} A_k(w) \longrightarrow s \text{ as } w \longrightarrow \infty$$

we say that $\sum_{n=1}^{\infty} a_n$ is summable by the Riesz method of type λ and order k to sum s, and write

$$\sum_{n=1}^{\infty} a_n = s\ (R,\lambda,k) .$$

If

$$\int_0^w \left| t^{-(k-1)} A_{k-1}(t) - s \right| dt = o(w)$$

we say that $\sum_{n=1}^{\infty} a_n$ is strongly summable by the Riesz method of type λ and order k to sum s, and write

$$\sum_{n=1}^{\infty} a_n = s\ [R,\lambda,k] .$$

If

$w^{-k} A_k(w)$ is of bounded variation with respect to w in the range $[\lambda_1,\infty)$, and $w^{-k} A_k(w) \longrightarrow s$ as $w \longrightarrow \infty$,

we say that $\sum_{n=1}^{\infty} a_n$ is absolutely summable by the Riesz method of type λ and order k to sum s, and write

$$\sum_{n=1}^{\infty} a_n = s\ |R,\lambda,k| .$$

The above definition of strong Riesz summability is that given by Srivastava [20] . Other definitions are

(i) if $\lambda_n = n$, if $w^{-k} A_k(w) \longrightarrow s$ as $w \longrightarrow \infty$ and if

$$\int_1^w \left| t \frac{d}{dt} t^{-k} A_k(t) \right|^r dt = o(w)$$

then $\sum_{n=1}^{\infty} a_n$ is said to be strongly summable by the Riesz method of type n and order k with index r to sum s , and is written

$$\sum_{n=1}^{\infty} a_n = s \, [R;k,r] ,$$

it being assumed that $k > 0$ and $r \geq 1$;

(This was given by Boyd and Hyslop [4])

and

(ii) if $\int_0^w \left| A_k(t) - s\, t^k \right|^r dt = o(w^{kr+1})$

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[20] P.Srivastava, "On strong Rieszian summability of infinite series", Proc. Nat. Inst. Sci. India, 23, A, (1957), 58 - 71 .

[4] A.V.Boyd and J.M.Hyslop, "A definition of strong Rieszian summability and its relationship to strong Cesaro summability", Proc. Glasgow Math. Assoc., 1, (1952 - 52) 94 - 99 .

then $\sum_{n=1}^{\infty} a_n$ is said to be strongly summable by the Riesz method of type λ and order k with index r to sum s, and is written

$$\sum_{n=1}^{\infty} a_n = s \; |R, \lambda, k|^r ,$$

it being assumed that $kr > -1$ and $0 < r < \infty$.
(This was given by Glatfield [9])

Boyd and Hyslop's definition is, for the case $r = 1$, equivalent to the one we use, but Glatfield's is not. In fact, Glatfield has shown that, for $r \geq 1$ and $kr > -1$, the methods $|R, n, k|^r$ and $[R; k+1, r]$ are equivalent. See [9], theorem 7.

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[9] M.Glatfield, "On strong Rieszian summability", Proc. Glasgow Math. Assoc., 3, (1956-58), 123 - 131 .

The following seven results are known for all positive k .

Theorem 1.1.1 .

For every $\delta \geq 0$, $\sum_{n=1}^{\infty} a_n = s\,(R,\lambda,k+\delta)$, *whenever* $\sum_{n=1}^{\infty} a_n = s\,(R,\lambda,k)$.

This result is due to Hardy and Riesz [14] .

Theorem 1.1.2 .

For every $\delta \geq 0$, $\sum_{n=1}^{\infty} a_n = s\,|R,\lambda,k+\delta|$, *whenever* $\sum_{n=1}^{\infty} a_n = s\,|R,\lambda,k|$.

This result is due to Obrechkoff [17] .

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[14] G.H.Hardy and M.Riesz, The general theory of Dirichlet series, Cambridge Tract No. 18, ist Ed., (1915) .

[17] N.Obrechkoff, "Sur la sommation absolue des series de Dirichlet", C.R. Acad. Sci. (Paris), 186, (1928), 215 - 217 .

Theorem 1.1.3 .

For every $\delta \geq 0$, $\sum_{n=1}^{\infty} a_n = s\,[R,\lambda,k+\delta]$, *whenever*

$\sum_{n=1}^{\infty} a_n = s\,[R,\lambda,k]$.

Theorem 1.2.1 .

$\sum_{n=1}^{\infty} a_n = s\,(R,\lambda,k)$ *whenever* $\sum_{n=1}^{\infty} a_n = s\,[R,\lambda,k]$.

Theorem 1.2.2 .

$\sum_{n=1}^{\infty} a_n = s\,[R,\lambda,k]$ *whenever* $\sum_{n=1}^{\infty} a_n = s\,|R,\lambda,k|$.

These last three results are all due to Srivastava [20] .

Theorem 1.3.1 .

$\sum_{n=1}^{\infty} a_n (\lambda_n)^{-k}$ *is summable* (R,e^{λ},k) *whenever* $\sum_{n=1}^{\infty} a_n$ *is summable* (R,λ,k) .

This result is due to Hardy [14] .

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[20] P.Srivastava, "On strong Rieszian summability of infinite series", Proc. Nat. Inst. Sci. India, 23, A, (1957), 58 - 71 .

[14] G.H.Hardy and M.Riesz, The general theory of Dirichlet series, Cambridge Tract No., 18, 1st Ed., (1915) .

Theorem 1.3.2 .

$\sum_{n=1}^{\infty} a_n (\lambda_n)^{-k}$ is summable $|R, e^{\lambda}, k|$ whenever $\sum_{n=1}^{\infty} a_n$ is summable $|R, \lambda, k|$.

This result is due to Tatchell [22] .

Theorem 1.3.3 .

$\sum_{n=1}^{\infty} a_n (\lambda_n)^{-k+1/q'}$ is summable $[R, e^{\lambda}; k, q]$ whenever $\sum_{n=1}^{\infty} a_n$ is summable $[R, \lambda; k, q]$ where $q \geq 1$, $kq' > 1$ and $1/q + 1/q' = 1$.

Theorem 1.3.4 .

$\sum_{n=1}^{\infty} a_n (\lambda_n)^{-k}$ is summable $[R, e^{\lambda}; k, q]$ whenever $\sum_{n=1}^{\infty} a_n$ is summable $[R, \lambda, k]$ to sum zero, where $q \geq 1$, $kq' > 1$ and $1/q + 1/q' = 1$ provided $\int_0^w |A_{k-1}(x)|^q \, dx = o(w^{kq})$.

These last two results are due to Srivastava [21] .

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[22] J.B.Tatchell, "A theorem on absolute Riesz summability", Journal London Math. Soc., 29, (1954), 49 - 59 .

[21] P.Srivastava, "Theorems on strong Riesz summability", Quart. Journal Math., (2), 11, (1960), 229 - 240 .

We have

$$A_k(w) = \sum_{\lambda_n < w} (w - \lambda_n)^k a_n = \int_0^w (w-t)^k \, dA(t) \quad ,$$

and define

$$B_k(w) = \sum_{\lambda_n < w} (w - \lambda_n)^k \lambda_n a_n = \int_0^w (w-t)^k \, t \, dA(t) \quad ,$$

$$F_k(w) = \sum_{\phi(\lambda_n) < w} \{w - \phi(\lambda_n)\}^k \psi(\lambda_n) a_n \quad ,$$

$$\text{Б}_k(w) = \sum_{\phi(\lambda_n) < w} \{w - \phi(\lambda_n)\}^k \phi(\lambda_n) \psi(\lambda_n) a_n \quad ,$$

$$G_k(w) = F_k\{\phi(w)\}$$

$$= \int_0^w \{\phi(w) - \phi(t)\}^k \psi(t) \, dA(t) \quad ,$$

and

$$\text{И}_k(w) = \text{Б}_k\{\phi(w)\}$$

$$= \int_0^w \{\phi(w) - \phi(t)\}^k \phi(t) \psi(t) \, dA(t) \quad .$$

Theorem 1.4 .

Suppose that $\mathfrak{W}_n(w) = \frac{1}{\Gamma(n)} \int_0^w (w-t)^{n-1} \mathfrak{w}(t)\, dt \quad (n > 0)$.

If

(i) $-1 < \alpha < 0 \quad ; \quad \beta \geq \alpha$,

(ii) $\mathfrak{W}_{\alpha+1}(w)$ is of bounded variation with respect to w in the range $(0, \xi)$,

(iii) $\mathfrak{W}_{\alpha+1}(0+) = 0$,

then

(a) $\mathfrak{W}_{\beta+1}(w)$ is a Lebesgue integral ,

(b) $\mathfrak{W}_{\beta+1}(0+) = 0$,

(c) for almost all w in $(0, \xi)$,

$$\mathfrak{W}_\beta(w) = \frac{1}{\Gamma(\beta-\alpha)} \int_0^w (w-t)^{\beta-\alpha-1}\, d\mathfrak{W}_{\alpha+1}(w) \quad .$$

This result is due to Bosanquet [3] .

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[3] L.S.Bosanquet, "Some extensions of Young's criterion for the convergence of a Fourier series", Quart. Journal Math., 6, (1935), 113 - 123 .

In view of this result, we can deduce :-

<u>Corollary 1.4.1 .</u>

$A_k(w)$ <u>is absolutely continuous, and for almost all</u> $w > 0$;

$$A_{k-1}(w) = \frac{1}{k} \frac{d}{dw} A_k(w) ,$$

and

<u>Corollary 1.4.2 .</u>

$B_k(w)$ <u>is absolutely continuous, and for almost all</u> $w > 0$;

$$B_{k-1}(w) = \frac{1}{k} \frac{d}{dw} B_k(w) .$$

Let $\eta(t)$ be Lebesgue integrable in every interval $[0,W]$. Fractional derivatives and integrals are defined as follows :-

$$I^k \eta(t) = \frac{1}{\Gamma(k)} \int_0^t (t-u)^{k-1} \eta(u)\, du ,$$

$${}_w I^k \eta(t) = \frac{1}{\Gamma(k)} \int_t^w (u-t)^{k-1} \eta(u)\, du ,$$

$$D^k \eta(t) = \frac{1}{\Gamma(p+1-k)} \left(\frac{d}{dt}\right)^{p+1} \int_0^t (t-u)^{p-k} \eta(u)\, du ,$$

and

$${}_wD_t^k \eta(t) = \frac{(-1)^{p+1}}{\Gamma(p+1-k)} \left(\frac{\partial}{\partial t}\right)^{p+1} \int_t^w (u-t)^{p-k} \eta(u)\, du .$$

The above derivatives are only defined when k is non-integral ; when k is a positive integer, we define

$${}_wD_t^k \eta(t) = (-1)^k D^k \eta(t) = (-1)^k \eta^{(k)}(t) .$$

We also have the following results :-

<u>Theorem 1.5 .</u>

$${}_wI^k\ {}_wD_t^k \left(\left\{1 - \frac{\phi(t)}{\phi(w)}\right\}^k \psi(t)\right) = \left\{1 - \frac{\phi(t)}{\phi(w)}\right\}^k \psi(t) .$$

<u>Proof.</u> The result can be easily verified if k is a positive integer. Suppose, therefore, from now on, that k is any positive non-integral numbers. Then

$${}_wD_t^k \left(\left\{1 - \frac{\phi(t)}{\phi(w)}\right\}^k \psi(t)\right)$$

$$= \frac{(-1)^{p+1}}{\Gamma(p+1-k)} \left(\frac{\partial}{\partial t}\right)^{p+1} \int_t^w (u-t)^{p-k} \left\{1 - \frac{\phi(u)}{\phi(w)}\right\}^k \psi(u) \quad du$$

$$= \frac{(-1)^p}{\Gamma(p+2-k)} \left(\frac{\partial}{\partial t}\right)^p \frac{\partial}{\partial t} \int_t^w (u-t)^{p+1-k} \frac{\partial}{\partial u}\left(\left\{1 - \frac{\phi(u)}{\phi(w)}\right\}^k \psi(u) \right) du$$

$$= \frac{(-1)^{p+1}}{\Gamma(p+1-k)} \left(\frac{\partial}{\partial t}\right)^p \int_t^w (u-t)^{p-k} \frac{\partial}{\partial u}\left(\left\{1 - \frac{\phi(u)}{\phi(w)}\right\}^k \psi(u) \right) du \ .$$

After p more applications of this same argument, we obtain

$${}_w D_t^k \left\{1 - \frac{\phi(t)}{\phi(w)}\right\}^k \psi(t)$$

$$= \frac{(-1)^{p+1}}{\Gamma(p+1-k)} \int_t^w (u-t)^{p-k} \left(\frac{\partial}{\partial u}\right)^{p+1} \left(\left\{1 - \frac{\phi(u)}{\phi(w)}\right\}^k \psi(u) \right) du \quad .(1.2)$$

Hence,

$${}_w I^k \ {}_w D_t^k \left\{1 - \frac{\phi(t)}{\phi(w)}\right\}^k \psi(t)$$

$$= \frac{(-1)^{p+1}}{\Gamma(k)\,\Gamma(p+1-k)} \int_t^w (v-t)^{k-1}\, dv \int_v^w (u-v)^{p-k} \ .$$

$$\cdot \left(\frac{\partial}{\partial u}\right)^{p+1} \left(\left\{1 - \frac{\phi(u)}{\phi(w)}\right\}^k \psi(u) \right) du$$

$$= \frac{(-1)^{p+1}}{\Gamma(k)\,\Gamma(p+1-k)} \int_t^w \left(\frac{\partial}{\partial u}\right)^{p+1} \left(\left\{1 - \frac{\phi(u)}{\phi(w)}\right\}^k \psi(u) \right) du \ .$$

$$\cdot \int_t^u (u-v)^{p-k} (v-t)^{k-1} \, dv$$

$$= \frac{(-1)^{p+1}}{\Gamma(p+1)} \int_t^w (u-t)^p \left(\frac{\partial}{\partial u}\right)^{p+1} \left(\left\{1 - \frac{\phi(u)}{\phi(w)}\right\}^k \psi(u) \right) du$$

$$= \left\{1 - \frac{\phi(t)}{\phi(w)}\right\}^k \psi(t) \quad ,$$

after $p+1$ integrations by parts.

This result is similar to lemma 5 in [11] in which Guha considers the case in which $\psi(t) = 1$.

<u>Theorem 1.6 .</u>

$$\int_0^w \left\{1 - \frac{\phi(t)}{\phi(w)}\right\}^k \psi(t) \quad dA(t)$$

$$= \frac{1}{\Gamma(k+1)} \int_0^w {}_wD_t^k \left(\left\{1 - \frac{\phi(t)}{\phi(w)}\right\}^k \psi(t) \right) \frac{d}{dt} A_k(t) \quad dt \quad .$$

<u>Proof.</u>

In view of theorem 1.5 ,

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[11] <u>U.C.Guha</u>, "The 'second theorem of consistency' for absolute Riesz summability", <u>Journal London Math. Soc.</u>, 31, (1956), 300 - 311 .

$$\int_0^w \left\{1 - \frac{\phi(t)}{\phi(w)}\right\}^k \psi(t) \quad dA(t)$$

$$= \int_0^w dA(t) \quad \frac{1}{\Gamma(k)} \int_t^w (v-t)^{k-1} \; {}_wD_v^k\left(\left\{1 - \frac{\phi(v)}{\phi(w)}\right\}^k \psi(v)\right) dv$$

$$= \int_0^w {}_wD_v^k\left(\left\{1 - \frac{\phi(v)}{\phi(w)}\right\}^k \psi(v)\right) dv \quad \frac{1}{\Gamma(k)} \int_0^v (v-t)^{k-1} \, dA(t) \quad .$$

Now, in view of corollary 1.4.1 , we have that $A_k(x)$ is absolutely continuous, and, for almost all $x > 0$,

$$\int_0^x (x-t)^{k-1} \, dA(t) = \frac{1}{k} \quad \frac{d}{dx} A_k(x) \quad .$$

Hence

$$\int_0^w \left\{1 - \frac{\phi(t)}{\phi(w)}\right\}^k \psi(t) \quad dA(t)$$

$$= \frac{1}{\Gamma(k+1)} \int_0^w {}_wD_t^k\left(\left\{1 - \frac{\phi(t)}{\phi(w)}\right\}^k \psi(t)\right) \frac{d}{dt} A_k(t) \quad dt \quad .$$

This result is similar to lemma 6 in [11] in which Guha considers the case in which $\psi(t) = 1$.

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[11] U.C.Guha, "The 'second theorem of consistency' for absolute Riesz summability", Journal London Math. Soc., 31, (1956), 300 - 311 .

We introduce the notation of "order of increase".

Suppose that ϕ_1, ϕ_2 are functions of a continuous variable t, defined for all values of $t \geq 0$. Suppose, also, that ϕ_1, ϕ_2 are positive, continuous and unboundedly increasing. We consider the ratio ϕ_1 / ϕ_2. There are four cases to consider :-

(i) if $\phi_1 / \phi_2 \to \infty$ as $t \to \infty$,

we write $\phi_1 \succ \phi_2$,

(ii) if $\phi_1 / \phi_2 \to 0$ as $t \to \infty$,

we write $\phi_1 \prec \phi_2$,

(iii) if, for all values of t, there are two positive numbers δ, Δ, such that $0 < \delta \leqslant \phi_1 / \phi_2 < \Delta$,

we write $\phi_1 \asymp \phi_2$,

if, however, ϕ_1 / ϕ_2 actually tends to a definite limit,

we write $\phi_1 \overset{=}{\asymp} \phi_2$,

and if this limit is unity,

we write $\phi_1 \sim \phi_2$,

and

(iv) if ϕ_1 / ϕ_2 neither tends to infinity, nor to zero, nor remains between fixed limits .

If a positive constant δ can be found such that $\phi_1 > \delta \phi_2$, for all sufficiently large values of t , we shall write $\phi_1 \succcurlyeq \phi_2$,

and if a positive constant Δ can be found such that $\phi_1 < \Delta \phi_2$, for all sufficiently large values of t , we shall write $\phi_1 \preccurlyeq \phi_2$.

For further properties of the above notation, see [13] .

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[13] G.H.Hardy, Orders of Infinity, Cambridge Tract No. 12, 1st Ed., (1910) .

A logarithmico-exponential function is defined to be a real single-valued function, defined for all values of the continuous variable t greater than some definite value, by a finite combination of the ordinary algebraical symbols (viz. $+$, $-$, $\times$, $\div$, $\sqrt[n]{\ }$) and the functional symbols $\log(\ldots)$ and $e^{(\ldots)}$, operating on the variable t and on real constants. It is to be observed that the result of working out the value of the function, by substituting t in the formula defining it, is to be real at all stages of the work. It is important to exclude functions such as

$$\tfrac{1}{2}\left\{e^{\sqrt{(-t^2)}} + e^{-\sqrt{(-t^2)}}\right\}$$

which, with suitable interpretation of the roots, is equal to $\cos t$.

Some useful properties of logarithmico-exponential functions are :-

(i) any logarithmico-exponential function is ultimately continuous, of constant sign, and monotonic, and, as $t \to \infty$, tends to infinity, or to zero, or to some definite limit,

(ii) if ϕ_1, ϕ_2 are logarithmico-exponential functions, then one of the relations

$$\phi_1 \succ \phi_2 \ ; \ \phi_1 \asymp \phi_2 \ ; \ \phi_1 \prec \phi_2$$

holds between them,

(iii) if ϕ_1 is a logarithmico-exponential function and $\phi_1(t) \succ e^t$, then there exists a positive integer n such that

$$e_n(t) \prec \phi_1(t) \preccurlyeq e_{n+1}(t) \ ,$$

where $e_n(t) = \exp\{e_{n-1}(t)\}$ and $e_0(t) = t$,

and

(iv) if ϕ_1, ϕ_2 are logarithmico-exponential functions not tending to finite limits, and $\phi_1 \preccurlyeq \phi_2$, then $\phi_1' \preccurlyeq \phi_2'$.

For further properties of logarithmico-exponential functions, see [13].

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[13] G.H.Hardy, Orders of Infinity, Cambridge Tract No. 12, 1st Ed., (1910).

We also have the following theorems for all k :-

Theorem 1.7.1 .

If (i) $\phi(w)$ is a logarithmico-exponential function,

(ii) $\phi(w) = O(w^{\delta})$, ($\delta > 0$) ,

then $\sum_{n=1}^{\infty} a_n$ is summable $(R, \phi(\lambda), k)$ whenever $\sum_{n=1}^{\infty} a_n$ is summable (R, λ, k) .

This result is due to Hardy. See [14] .

Theorem 1.7.2 .

If (i) $\phi(w)$ is a logarithmico-exponential function,

(ii) $\phi(w) = O(w^{\delta})$, ($\delta > 0$) ,

then $\sum_{n=1}^{\infty} a_n$ is summable $|R, \phi(\lambda), k|$ whenever $\sum_{n=1}^{\infty} a_n$ is summable $|R, \lambda, k|$.

This result is due to Chandrasekharan. See [6] .

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[14] G.H.Hardy and M.Riesz, The General Theory of Dirichlet Series, Cambridge Tract No. 18, 1st Ed., (1915) .

[6] K.Chandrasekharan, "The second theorem of consistency for absolutely summable series", Journal Indian Math. Soc., (2), 6, (1942), 168 - 180 .

1.3 Statement of the problem.

The general problem considered in this thesis is that of finding sufficient conditions on the functions $\phi(w)$, $\psi(w)$, such that $\sum_{n=1}^{\infty} a_n \psi(\lambda_n)$ is ordinarily, absolutely or strongly summable by the Riesz method of type $\phi(\lambda)$ and order k , according as $\sum_{n=1}^{\infty} a_n$ is ordinarily, absolutely or strongly summable by the Riesz method of type λ and order k .

For ordinary and absolute summability, we consider general functions $\phi(w)$, $\psi(w)$, as defined earlier, but for strong summability, the particular case in which $\phi(w)$ is a logarithmico-exponential function and

$$\psi(w) = \left\{\frac{\phi(w)}{w\ \phi'(w)}\right\}^k .$$

Chapter II.

Ordinary Riesz Summability.

2.1 Introduction.

In this chapter, we seek conditions, sufficient for the truth of the proposition

P_1 : $\sum_{n=1}^{\infty} a_n \psi(\lambda_n)$ <u>is summable</u> $(R, \phi(\lambda), k)$ <u>whenever</u>

$\sum_{n=1}^{\infty} a_n$ <u>is summable</u> (R, λ, k) .

When k is a positive integer, the following theorem has been proved by Borwein [1] .

Theorem 2.1 .

<u>If</u> (i) $\gamma(w)$ <u>is positive and absolutely continuous in every interval</u> $[a, W]$, <u>and</u> $\gamma'(w) = O(1)$ <u>for</u> $w \geq a$,

(ii) $w^n \psi^{(n)}(w) = O(\{\gamma(w) / w\}^{k-n})$ <u>for</u> $w \geq a$ <u>and</u> $n = 0, 1, \ldots, k$,

(iii) $\int_a^{\infty} t^k \left|\psi^{(k+1)}(t)\right| dt < \infty$,

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[1] <u>D.Borwein</u>, "A theorem on Riesz summability", <u>Journal London Math. Soc.</u>, 31, (1956), 319 - 324 .

(iv) $\int_a^w \{\gamma(t)\}^n \, |\phi^{(n+1)}(t)| \, dt = O\{\phi(w)\}$ <u>for</u> $w \geq a$

<u>and</u> $n = 0, 1, \ldots, k$,

<u>then</u> P_1 .

Other known theorems which hold for all $k \geq 0$ are

<u>Theorem 2.2 .</u>

<u>If</u> $\phi(w) = e^w$ <u>and</u> $\psi(w) = w^{-k}$, <u>then</u> P_1 ,

(This is a re-statement of theorem 1.3.1)

<u>Theorem 2.3 .</u>

<u>If</u> (i) $\phi(w)$ <u>is a logarithmico-exponential function</u> ,

(ii) $\frac{1}{w} \prec \frac{\phi'(w)}{\phi(w)} \prec 1$,

(iii) $\psi(w) = \left\{\frac{\phi(w)}{w\,\phi'(w)}\right\}^k$,

<u>then</u> P_1 ,

and

<u>Theorem 2.4 .</u>

<u>If</u> (i) $\phi(w)$ <u>is a logarithmico-exponential function</u> ,

(ii) $\frac{1}{w} \preccurlyeq \frac{\phi'(w)}{\phi(w)}$,

(iii) $\psi(w) = \left\{ \dfrac{\phi(w)}{w\ \phi'(w)} \right\}^k$,

then P_1 .

Theorem 2.2 , which is included in theorem 2.4 , is a well-known theorem of Hardy [14] , and theorems 2.3 and 2.4 are due to Guha [10] , who derived the latter from the former by means of standard results. For integral values of k , the hypotheses of theorem 2.1 are satisfied with $\phi(w)$, $\psi(w)$ as in theorem 2.4 , and $\gamma(w) = \phi(w) / \phi'(w)$.

We prove the following theorems as companions to theorem 2.1 .

Theorem 2.5 .

Suppose that k *is any positive non-integral number.*

If (i) $\gamma(w)$ *is positive and absolutely continuous in every interval* $[a,W]$, *and* $\gamma'(w) = O(1)$ *for* $w \geq a$,

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[14] G.H.Hardy and M.Riesz, The General Theory of Dirichlet Series, Cambridge Tract No. 18, 1st Ed., (1915) .

[10] U.C.Guha, "Convergence factors for Riesz summability", Journal London Math. Soc., 31, (1956), 311 - 319 .

(ii) (a) $\psi(w) = O(\{\gamma(w) / w\}^k)$ *for* $w \geq a$,

(b) $w^n \psi^{(n)}(w) = O(\{\gamma(w) / w\}^{p+1-n})$ *for* $w \geq a$ *and* $n = 1, 2, \ldots, p+1$,

(iii) $\int_a^\infty t^{p+1} |\psi^{(p+2)}(t)| \, dt < \infty$,

(iv) $\phi'(w)$ *is positive monotonic non-decreasing for* $w \geq a$,

(v) $\gamma(w)\, \phi'(w) = O\{\phi(w)\}$ *for* $w \geq a$ *or* $\{\gamma(w)\}^{n-1} \phi^{(n)}(w) \{\phi'(w)\}^{-1}$ *is of bounded variation with respect to* w *in the range* $[a, \infty)$ *for* $n = 1, 2, \ldots, p+1$, *according as* $0 < k < 1$ *or* $k > 1$,

(vi) $\phi''(w) / \phi'(w)$ *is monotonic non-increasing for* $w \geq a$,

(vii) $h_n(w) = \psi(w) \{\phi'(w)\}^{k-n} \{\gamma(w)\}^{-n}$ *is positive monotonic in the range* $w \geq a$ *for* $n = 0, 1, \ldots, p$, *possibly in different senses for different values of* n ,

(viii) $\phi(w) > M\, w^{k/(k-p)}$ *for* $w \geq a$,

then P_1 .

Theorem 2.6 .

Suppose that k is any positive non-integral number.
If conditions (i) to (vii) inclusive of theorem 2.5 hold, and, in addition,

(vii)' $h_p(w)$ is non-decreasing for $w \geq a$,

then P_1 .

It is evident that theorem 2.2 , for non-integral k , is included in theorem 2.5 , and it can readily be shown that, under the hypotheses of theorem 2.3 , the hypotheses of theorem 2.5 are satisfied with $\gamma(w) = \phi(w) / \phi'(w)$ and $\phi(w)$, $\psi(w)$ as in theorem 2.3 .

Originally, the theorem obtained was :-

Theorem 2.5.0 .

Suppose that k is any positive non-integral number.
If conditions (i) to (vi) inclusive of theorem 2.5 hold, and, in addition,

(vii)" $h_n(w) = \psi(w) \{\phi'(w)\}^{k-n} \{\gamma(w)\}^{-n}$ is positive monotonic non-decreasing in the range $w \geq a$ for $n = 0, 1, \ldots, p$,

then P_1 .

The present formulation of the results (viz. theorems 2.5 and 2.6) arose from valuable suggestions by the referee appointed by the Glasgow Mathematical Association when theorem 2.5.0 was submitted for publication. The argument in section 2.4 is due to the referee : it shows that the conditions of theorem 2.6 are in fact more stringent than those of theorem 2.5 .

2.2 Lemmas.

The following lemmas are required :-

Lemma 2.1 .

The n^{th} derivative of $\{f(t)\}^m$ is a sum of a number of terms like

$$M_1 \ \{f(t)\}^{m-\sigma} \ \prod_{\nu=1}^{n} \{f^{(\nu)}(t)\}^{\alpha_\nu}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are non-negative integers such that

$$1 \leq \sum_{\nu=1}^{n} \alpha_\nu = \sigma \leq \sum_{\nu=1}^{n} \nu\alpha_\nu = n .$$

If m is a positive integer, then $\sigma \leq m$.

This simple result is a particular case of a theorem due to Faa di Bruno. See [23] , pp. 88 - 89 .

Lemma 2.2 .

If (i) $\gamma(w)$ *is positive and absolutely continuous in every interval* $[a, W]$, *and* $\gamma'(w) = O(1)$ *for* $w \geq a$,

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[23] C-J. de la Vallee Poussin, Cours D'analyse Infinitesimale, Louvain : Paris, 4th Ed., (1921 - 22) .

(ii) $\gamma(w)\,\phi'(w) = O\{\phi(w)\}$ _for_ $w \geq a$ _or_ $\{\gamma(w)\}^{n-1}\,\phi^{(n)}(w)\,\{\phi'(w)\}^{-1}$ _is of bounded variation with respect to_ w _in the range_ $[a, \infty)$ _for_ $n = 1, 2, \ldots, p+1$ _according as_ $0 < k < 1$ _or_ $k > 1$,

then, for $n = 1, 2, \ldots, p+1$ _and_ $w \geq a$,

$$\text{(a)} \quad \int_a^w \{\gamma(t)\}^{n-1}\,|\phi^{(n)}(t)|\,dt = O\{\phi(w)\} ,$$

and

$$\text{(b)} \quad \{\gamma(w)\}^n\,\phi^{(n)}(w) = O\{\phi(w)\} .$$

Proof. When $0 < k < 1$, (b) is the same as (ii) , and (a) is a trivial consequence. Suppose, therefore, that $k > 1$. Then (a) follows from the appropriate part of (ii) by integration : hence

$$\gamma(w)\,\phi'(w) = \gamma(a)\,\phi'(a) + \int_a^w \gamma(t)\,\phi''(t)\,dt + \int_a^w \gamma'(t)\,\phi'(t)\,dt$$

$$= O\{\phi(w)\} ,$$

since $\gamma'(t) = O(1)$, and (b) is an immediate consequence. Compare [1] , lemma 2 .

<u>Lemma 2.3 .</u>

If $\overline{\lim}_{w \to \infty} \int_a^\infty |f(w,t)| \, dt < \infty$ *and*

$$\lim_{w \to \infty} \int_a^y |f(w,t)| \, dt = 0 \quad \textit{for every finite } y \geq a ,$$

and if $s(t)$ *is a bounded measurable function in* $[a,\infty)$ *which tends to zero as* t *tends to infinity, then*

$$\lim_{w \to \infty} \int_a^\infty f(w,t) \, s(t) \, dt = 0 .$$

<u>Proof.</u>

$$\overline{\lim}_{w \to \infty} \left| \int_a^\infty f(w,t) \, s(t) \, dt \right|$$

$$\leq \overline{\lim}_{w \to \infty} \int_a^y |f(w,t) \, s(t)| \, dt + \overline{\lim}_{w \to \infty} \int_y^\infty |f(w,t) \, s(t)| \, dt$$

$$< \overline{\underset{t \geq y}{\mathrm{bd}}} \, |s(t)| \, . \, \overline{\lim}_{w \to \infty} \int_a^\infty |f(w,t)| \, dt \, .$$

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[1] D.Borwein, "A theorem on Riesz summability", Journal London Math. Soc., 31, (1956), 319 - 324 .

Since the final expression tends to zero as y tends to infinity, the result follows. Compare [1], lemma 3, or [12], p. 50.

Lemma 2.4.

If (i) $\phi'(w)$ *is positive monotonic non-decreasing for* $w \geq a$,

(ii) $\phi''(w) / \phi'(w)$ *is monotonic non-increasing for* $w \geq a$,

then $X(t) = \dfrac{1}{\phi'(t)} \cdot \dfrac{\phi(w) - \phi(t)}{w - t}$ *is a monotonic non-increasing function of* t *for* $a \leq t < w$.

Proof.

We have, for $a \leq t < w$,

$$\frac{X'(t)}{X(t)} = \frac{\{\phi(w) - \phi(t)\} - (w-t)\,\phi'(t)}{\{\phi(w) - \phi(t)\}\,(w-t)} - \frac{\phi''(t)}{\phi'(t)}$$

$$= \frac{\phi'(\eta) - \phi'(t)}{\phi(w) - \phi(t)} - \frac{\phi''(t)}{\phi'(t)} \qquad \text{where } w > \eta > t$$

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[1] D. Borwein, "A theorem on Riesz summability", Journal London Math. Soc., 31, (1956) 319 - 324.

[12] G.H. Hardy, Divergent Series, Oxford, (1949).

$$\leq \quad \frac{\phi'(w) - \phi'(t)}{\phi(w) - \phi(t)} - \frac{\phi''(t)}{\phi'(t)}$$

$$= \quad \frac{\phi''(\xi)}{\phi'(\xi)} - \frac{\phi''(t)}{\phi'(t)} \qquad \text{where } w > \xi > t$$

$$\leq \quad 0 \; .$$

Since $X(t) \geq 0$, the result follows.

<u>Lemma 2.5 .</u>

<u>If</u> a_n <u>is</u> <u>real</u> <u>and</u> $a < \xi < w$, <u>then</u>

$$\frac{\Gamma(k+1)}{\Gamma(p+1)\,\Gamma(k-p)} \left| \int_a^\xi A_p(t)\,(w-t)^{k-p-1}\,dt \right| \leq \max_{a<t<\xi} |A_k(t)|$$

<u>Proof.</u>

$$\int_a^\xi A_p(t)\,(w-t)^{k-p-1}\,dt$$

$$= \frac{\Gamma(p+1)}{\Gamma(k+1)\,\Gamma(p+1-k)} \int_a^\xi (w-t)^{k-p-1}\,dt \int_a^t \frac{d}{du} A_k(u)\,(t-u)^{p-k}\,du$$

$$= \int_a^\xi \frac{d}{du} A_k(u) \quad h(u) \quad du$$

where

$$h(u) = \frac{\Gamma(p+1)}{\Gamma(k+1)\,\Gamma(p+1-k)} \int_u^\xi (w-t)^{k-p-1}\,(t-u)^{p-k}\,dt$$

$$= \frac{\Gamma(p+1)\,\Gamma(k-p)}{\Gamma(k+1)} - \frac{\Gamma(p+1)}{\Gamma(k+1)\,\Gamma(p+1-k)} \;.$$

$$\cdot \int_{\xi}^{w} (w-t)^{k-p-1}\,(t-u)^{p-k}\,dt \;.$$

Now, if t has any fixed value between ξ and w, and $0 \leq u \leq \xi$, then $(t-u)^{p-k}$ increases with u. Hence $h(u)$ is a positive monotonic non-increasing function of u. Applying the second mean value theorem for integrals, we obtain that

$$\int_{0}^{\xi} A_p(t)\,(w-t)^{k-p-1}\,dt = h(0)\,A_k(\eta)$$

where $0 \leq \eta \leq \xi$, and this proves the lemma.

This result is due essentially to Riesz. Compare [14], lemma 8, or [18].

[14] G.H.Hardy and M.Riesz, The General Theory of Dirichlet Series, Cambridge Tract No. 18, 1st Ed., (1915).

[18] M.Riesz, "Sur un theoreme de la moyenne et ses applications", Acta Litt. ac Sci. Univ. Hungaricae (Szeged), 1, (1923), 114 - 126.

2.3 Proof of theorem 2.5 .

We must prove that, if $w^{-k} A_k(w)$ tends to a finite limit as w tends to infinity, then $w^{-k} F_k(w)$ tends to a finite limit as w tends to infinity. We assume, therefore, without loss of generality, that

$$A(w) = 0 \quad \text{for} \quad 0 \leq w \leq a ,$$

$$\text{and} \quad A_k(w) = o(w^k) , \tag{2.3.1}$$

and note that it is sufficient to prove that

$$\{\phi(w)\}^{-k} G_k(w) = \{\phi(w)\}^{-k} \int_a^w \{\phi(w) - \phi(t)\}^k \psi(t)\, dA(t)$$

tends to a finite limit as w tends to infinity.

After $p+1$ integrations by parts, $\{\phi(w)\}^{-k} G_k(w)$ reduces to a constant multiple of

$$\int_a^w A_p(t) \left(\frac{\partial}{\partial t}\right)^{p+1} \left(\left\{1 - \frac{\phi(t)}{\phi(w)}\right\}^k \psi(t)\right) dt$$

which, by lemma 2.1 and Leibnitz's theorem on the differentiation of a product, can be expressed as a sum of constant multiples of integrals of the types :-

$$I_1 = \{\phi(w)\}^{-k} \int_a^w A_p(t)\ \psi^{(p+1)}(t)\ \{\phi(w) - \phi(t)\}^k\ dt\ ,$$

$$I_2 = \{\phi(w)\}^{-k} \int_a^w A_p(t)\ \psi^{(p+1-r)}(t)\ \{\phi(w) - \phi(t)\}^{k-\sigma}\ .$$

$$\cdot \prod_{\nu=1}^{r} \{\phi^{(\nu)}(t)\}^{\alpha_\nu}\ dt\ ,$$

and

$$I_3 = \{\phi(w)\}^{-k} \int_a^w A_p(t)\ \psi(t)\ \{\phi(w) - \phi(t)\}^{k-\mu}\ .$$

$$\cdot \prod_{\nu=1}^{p+1} \{\phi^{(\nu)}(t)\}^{\beta_\nu}\ dt\ ,$$

where $\alpha_1, \alpha_2, \ldots, \alpha_r, \beta_1, \beta_2, \ldots, \beta_{p+1}$ are non-negative integers such that

$$1 \leq \sum_{\nu=1}^{r} \alpha_\nu = \sigma \leq \sum_{\nu=1}^{r} \nu\alpha_\nu = r \leq p\ ,$$

and

$$1 \leq \sum_{\nu=1}^{p+1} \beta_\nu = \mu \leq \sum_{\nu=1}^{p+1} \nu\beta_\nu = p+1\ .$$

Consider first I_1. Integrate it by parts to obtain

$$(p+1)\ I_1 = -I_{11} + k\ I_{12}\ ,$$

where

$$I_{11} = \{\phi(w)\}^{-k} \int_a^w A_{p+1}(t)\ \psi^{(p+2)}(t)\ \{\phi(w) - \phi(t)\}^k \ dt ,$$

and

$$I_{12} = \{\phi(w)\}^{-k} \int_a^w A_{p+1}(t)\ \psi^{(p+1)}(t)\ \phi'(t)\ \{\phi(w) - \phi(t)\}^{k-1} \ dt .$$

Now, by theorem 1.1.1 , we obtain that $\sum_{n=1}^{\infty} a_n$ is summable $(R,\lambda,p+1)$ to sum zero, and hence that

$$A_{p+1}(w) = o(w^{p+1}) . \tag{2.3.2}$$

Hence, using this and condition (iii) of theorem 2.5 , we obtain that

$$\int_a^{\infty} \left| A_{p+1}(t)\ \psi^{(p+2)}(t) \right| \ dt < \infty ,$$

and so, by Lebesgue's theorem on dominated convergence (see e.g. [5] , p.41, theorem C) , I_{11} tends to

$$\theta = \int_a^{\infty} A_{p+1}(t)\ \psi^{(p+2)}(t) \ dt$$

as w tends to infinity, θ being finite.

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[5] J.C.Burkill, The Lebesgue Integral, Cambridge Tract No. 41, Ed., (1958) .

For I_{12}, consider the function

$$q_1(w,t) = \{\phi(w)\}^{-k} t^{p+1} \psi^{(p+1)}(t) \phi'(t) \{\phi(w) - \phi(t)\}^{k-1} .$$

Now, by condition (ii) of theorem 2.5, we note that for $w > t \geq a$,

$$|q_1(w,t)| < M \{\phi(w)\}^{-k} \phi'(t) \{\phi(w) - \phi(t)\}^{k-1} .$$

Hence $q_1(w,t)$ satisfies the hypotheses of lemma 2.3, and so

$$\int_a^w q_1(w,t)\, t^{-p-1} A_{p+1}(t) \, dt \longrightarrow 0$$

as w tends to infinity.

That is $\lim_{w\to\infty} I_{12} = 0$,

and so $\lim_{w\to\infty} I_1 = 0$. (2.3.3)

Considering now I_2, we see, on integrating by parts, that it is equal to a sum of constant multiples of integrals of the types

$$I_{21} = \{\phi(w)\}^{-k} \int_a^w A_{p+1}(t) \psi^{(p+2-r)}(t) \{\phi(w) - \phi(t)\}^{k-\sigma} .$$

$$. \prod_{\nu=1}^{r} \{\phi^{(\nu)}(t)\}^{\alpha_\nu} \, dt ,$$

$$I_{22} = \{\phi(w)\}^{-k} \int_a^w A_{p+1}(t)\ \psi^{(p+1-r)}(t)\ \{\phi(w) - \phi(t)\}^{k-\sigma-1}\ .$$

$$.\ \phi'(t)\ \prod_{\nu=1}^{r} \left\{\phi^{(\nu)}(t)\right\}^{\alpha_\nu}\ dt\ ,$$

and

$$I_{23} = \{\phi(w)\}^{-k} \int_a^w A_{p+1}(t)\ \psi^{(p+1-r)}(t)\ \{\phi(w) - \phi(t)\}^{k-\sigma}\ .$$

$$.\ \prod_{\nu=1}^{r+1} \left\{\phi^{(\nu)}(t)\right\}^{\delta_\nu}\ dt\ ,$$

where $\alpha_1, \alpha_2, \ldots, \alpha_r, \delta_1, \delta_2, \ldots, \delta_{r+1}$ are non-negative integers such that

$$1 \leqq \sum_{\nu=1}^{r} \alpha_\nu = \sigma \leqq \sum_{\nu=1}^{r} \nu\alpha_\nu = r \leqq p\ ;$$

$$\sum_{\nu=1}^{r+1} \delta_\nu = \sigma\ ; \quad \sum_{\nu=1}^{r+1} \nu\delta_\nu = r+1\ .$$

For I_{21}, consider the function

$$q_2(w,t) = \{\phi(w)\}^{-k}\ t^{p+1}\ \psi^{(p+2-r)}(t)\ \{\phi(w) - \phi(t)\}^{k-\sigma}\ .$$

$$.\ \prod_{\nu=1}^{r} \left\{\phi^{(\nu)}(t)\right\}^{\alpha_\nu}\ .$$

Suppose that the non-vanishing α_ν of highest suffix is α_s. Then

$$q_2(w,t) = \{\phi(w)\}^{-k} t^{p+1} \psi^{(p+2-r)}(t)\, \phi^{(s)}(t) \{\phi(w) - \phi(t)\}^{k-\sigma} .$$

$$. \left\{\phi^{(s)}(t)\right\}^{\alpha_s - 1} \prod_{\nu=1}^{s-1} \left\{\phi^{(\nu)}(t)\right\}^{\alpha_\nu} \quad dt$$

and

$$1 \leqq \sum_{\nu=1}^{s} \alpha_\nu = \sigma \leqq \sum_{\nu=1}^{s} \nu\alpha_\nu = r \leqq p .$$

Using lemma 2.2 (b) and condition (ii) of theorem 2.5, we find that, for $w > t \geqq a$,

$$\left|q_2(w,t)\right| < M \{\phi(w)\}^{-k} t^{p+1} \{\gamma(t)\}^{r-1} t^{-p-1} \left|\phi^{(s)}(t)\right| .$$

$$. \{\phi(w) - \phi(t)\}^{k-\sigma} \{\phi(t)\}^{\sigma-1} \{\gamma(t)\}^{s-r} ,$$

$$< M \{\phi(w)\}^{-1} \{\gamma(t)\}^{s-1} \left|\phi^{(s)}(t)\right| .$$

Hence, in view of lemma 2.2 (a), $q_2(w,t)$ satisfies the hypotheses of lemma 2.3, and so

$$\int_a^w q_2(w,t)\, t^{-p-1} A_{p+1}(t) \quad dt \rightarrow 0$$

as w tends to infinity.

That is $\lim_{w \to \infty} I_{21} = 0$.

Similarly $\lim_{w \to \infty} I_{23} = 0$,

and $\lim_{w \to \infty} I_{22} = 0$ in the cases where $k-1-\sigma > 0$.

The remaining case of I_{22} is that in which $r = \sigma = p$, and we write the remaining integral as

$$\{\phi(w)\}^{-k} \int_a^w A_{p+1}(t)\ \psi'(t)\ \{\phi'(t)\}^{p+1}\ \{\phi(w) - \phi(t)\}^{k-p-1}\ dt \ .$$

For this, consider the function

$$q_3(w,t) = \{\phi(w)\}^{-k}\ t^{p+1}\ \psi'(t)\ \phi'(t)\ \{\phi(w) - \phi(t)\}^{k-p-1} \ .$$
$$. \ \{\phi'(t)\}^p \ .$$

Using lemma 2.2 (b) and condition (ii) of theorem 2.5 , we find that, for $w > t \geq a$,

$$|q_3(w,t)| < M\ \{\phi(w)\}^{-k}\ t^{p+1}\ \{\gamma(t)\}^p\ t^{-p-1}\ \phi'(t) \ .$$
$$. \ \{\phi(w) - \phi(t)\}^{k-p-1}\ \{\phi(t)\}^p\ \{\gamma(t)\}^{-p} \ ,$$

$$< M\ \{\phi(w)\}^{p-k}\ \phi'(t)\ \{\phi(w) - \phi(t)\}^{k-p-1} \ .$$

Hence $q_3(w,t)$ satisfies the hypotheses of lemma 2.3 , and so

$$\int_a^w q_3(w,t)\ \ t^{-p-1}\ A_{p+1}(t)\ \ dt \longrightarrow 0$$

as w tends to infinity.

That is $\lim_{w \to \infty} I_{22} = 0$ in the case where $r = \sigma = p$.

Hence $\lim_{w \to \infty} I_2 = 0$ . . (2.3.4)

Finally, consider I_3 , which can be written in the form

$$I_3 = \{\phi(w)\}^{-k} \int_a^w A_p(t) \; (w-t)^{k-p-1} \{\phi(w) - \phi(t)\}^{p+1-\mu} \; g(t) \; .$$

$$. \; H(t) \; h_{p+1-\mu}(t) \quad dt$$

where

$$g(t) = \left(\frac{1}{\phi'(t)} \cdot \frac{\phi(w) - \phi(t)}{w - t} \right)^{k-p-1} \quad \text{for} \quad a \leq t < w \;,$$

$$g(w) = 1 \;,$$

and

$$H(t) = \prod_{\nu=1}^{p+1} \left(\frac{\{\gamma(t)\}^{\nu-1} \; \phi^{(\nu)}(t)}{\phi'(t)} \right)^{\beta_\nu}$$

where $\beta_1 , \beta_2 , \ldots , \beta_{p+1}$ are non-negative integers such that

$$1 \leq \sum_{\nu=1}^{p+1} \beta_\nu = \mu \leq \sum_{\nu=1}^{p+1} \nu\beta_\nu = p+1 \; .$$

Then $H(t)$ is of bounded variation with respect to t in the range $[a,\infty)$, because of condition (v) of theorem 2.5, and so can be expressed as the difference between two bounded monotonic non-increasing functions. Consequently, we can assume, without loss of generality, that $H(t)$ is bounded and monotonic non-increasing for $w \geqq a$. Also, $\{\phi(w) - \phi(t)\}^{p+1-\mu}$, $g(t)$ and $h_{p+1-\mu}(t)$ are monotonic functions of t in the range $[a,w]$, the first being non-increasing since $p+1-\mu \geq 0$, and the second, non-decreasing in view of lemma 2.4. Using the second mean value theorem for integrals twice, we now see that

$$I_3 = \{\phi(w)\}^{-k} \{\phi(w)\}^{p+1-\mu} H(a)\, g(w)\, h_{p+1-\mu}(x) \,.$$

$$. \int_{\xi_1}^{\xi_2} A_p(t)\, (w-t)^{k-p-1}\, dt$$

where $w \geq \xi_2 > \xi_1 \geq a$ and $x = w$ or a according as $h_{p+1-\mu}(t)$ is non-decreasing or non-increasing. Hence,

$$I_3 = o\left(\{\phi(w)\}^{p+1-k-\mu}\, w^k\, h_{p+1-\mu}(x)\right)$$

$$= o\left\{Q(w,x)\right\}, \quad \text{say,}$$

where

$$Q(w,x) = \{\phi(w)\}^{p+1-\mu-k} w^k \, h_{p+1-\mu}(x) \quad .$$

Now, in view of lemma 2.2 (b) and condition (ii) of theorem 2.5 ,

$$Q(w,w) = O\Big(\{\phi(w)\}^{p+1-k-\mu} \, w^k \, \psi(w) \, \{\gamma(w)\}^{\mu-p-1} \; . \; \{\phi'(w)\}^{k+\mu-p-1}\Big) \; ,$$

$$= O(1) \; ,$$

and, in view of condition (viii) of theorem 2.5 ,

$$Q(w,a) = O\Big(\{\phi(w)\}^{p+1-k-\mu} \, w^k\Big) \; ,$$

$$= O\Big(\{\phi(w)\}^{1-\mu}\Big) \; ,$$

$$= O(1)$$

since $\mu \geq 1$.

Hence $\lim\limits_{w \to \infty} I_3 = 0$. (2.3.5)

Hence, because of (2.3.3) , (2.3.4) and (2.3.5) , we can deduce that $\{\phi(w)\}^{-k} G_k(w)$ tends to a finite limit as w tends to infinity.

This completes the proof of theorem 2.5 .

2.4 Proof of theorem 2.6 .

Suppose that conditions (i) and (ii) (a) of theorem 2.5 and condition (vii)' of theorem 2.6 hold. It is clearly sufficient to show that condition (viii) of theorem 2.5 is a consequence : i.e. , we wish to show that, if

(2.4.1) $\gamma(w)$ is positive and absolutely continuous in every interval $[a, W]$, and $\gamma'(w) = O(1)$ for $w \geq a$,

(2.4.2) $\psi(w) = O\left(\{\gamma(w) / w\}^k\right)$ for $w \geq a$,

and

(2.4.3) $h_p(w)$ is monotonic non-decreasing for $w \geq a$,

then

(2.4.4) $\phi(w) > M\, w^{k/(k-p)}$ for $w \geq a$.

Now, it follows from (2.4.3) that, for $w \geq a$,

$$\frac{\psi(w) \{\phi'(w)\}^{k-p}}{\{\gamma(w)\}^p} > M ,$$

and hence, in view of (2.4.2) ,

$$\begin{aligned}\{\gamma(w)\}^p &= O\left(\{\psi(w)\}\{\phi'(w)\}^{k-p}\right) , \\ &= O\left(\{\gamma(w) / w\}^k \{\phi'(w)\}^{k-p}\right) .\end{aligned}$$

Consequently, by (2.4.1) ,

$$w^k = O\left(\{\gamma(w)\ \phi'(w)\}^{k-p}\right) ,$$

$$= O\left(\{w\ \phi'(w)\}^{k-p}\right) ,$$

and so

$$w^p = O\left(\{\phi'(w)\}^{k-p}\right) .$$

Hence, for $w \geqq a$,

$$\phi'(w) > M\, w^{p/(k-p)} ,$$

and hence, on integrating, we obtain (2.4.4) .

This completes the proof of theorem 2.6 .

Chapter III.

Absolute Riesz Summability.

3.1 Introduction.

In this chapter, we seek conditions sufficient for the truth of the proposition

P_2 : $\sum_{n=1}^{\infty} a_n \psi(\lambda_n)$ is summable $|R, \phi(\lambda), k|$ whenever

$\sum_{n=1}^{\infty} a_n$ is summable $|R, \lambda, k|$.

The following theorems are known :-

Theorem 3.1 .

For all $k > 0$, if $\phi(w) = e^w$ and $\psi(w) = w^{-k}$, then P_2 .

This is a re-statement of theorem 1.3.2 .

Theorem 3.2 .

Suppose that k is a positive integer.

If (i) $\gamma(w)$ is positive and absolutely continuous in every interval $[a, W]$ and $\gamma(w) / w$ is of bounded variation with respect to w in the range $[a, \infty)$,

(ii) $w^n \psi^{(n)}(w) = O\left(\{\gamma(w) / w\}^{k-n} \right)$ for $w \geq a$ and $n = 0, 1, \ldots, k$,

(iii) $\{\gamma(w)\}^n \phi^{(n)}(w) = O\{\phi(w)\}$ *for* $w \geq a$

and $n = 1, 2, \ldots, k$,

then P_2 ,

Theorem 3.3 .

Suppose that k *is a positive integer.*

If (i) $\phi(w)$ *is a logarithmico-exponential function* ,

(ii) $\frac{1}{w} \prec \frac{\phi'(w)}{\phi(w)} \prec 1$,

(iii) $\psi(w) = \left\{\frac{\phi(w)}{w\,\phi'(w)}\right\}^k$,

then P_2 ,

and

Theorem 3.4 .

Suppose that k *is a positive integer.*

If (i) $\phi(w)$ *is a logarithmico-exponential function* ,

(ii) $\frac{1}{w} \preccurlyeq \frac{\phi'(w)}{\phi(w)}$,

(iii) $\psi(w) = \left\{\frac{\phi(w)}{w\,\phi'(w)}\right\}^k$,

then P_2 .

Theorem 3.1 is a well-known theorem by Tatchell [21], theorem 3.2 is due to Dikshit [7] and theorems 3.3 and 3.4 are due to Guha [10], who derived the latter from the former by means of standard results.

We prove the following theorems :-

Theorem 3.5.

Suppose that k is a positive integer.

If (i) $\gamma(w)$ is positive and absolutely continuous in every interval $[a, W]$ and $\gamma(w) = O(w)$ for $w \geq a$,

(ii) $w^n \psi^{(n)}(w) = O\left(\{\gamma(w) / w\}^{k-n}\right)$ for $w \geq a$ and $n = 0, 1, \ldots, k$,

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[21] J.B.Tatchell, "A theorem on absolute Riesz summability", Journal London Math. Soc., 29, (1954), 49 - 59.

[7] G.D.Dikshit, "On the absolute Riesz summability factors of infinite series, I", Indian Journal Math., 1, (1958), 33 - 40.

[10] U.C.Guha, "Convergence factors for Riesz summability", Journal London Math. Soc., 31, (1956), 311 - 319.

(iii) $\{\gamma(w)\}^n \phi^{(n)}(w) = O\{\phi(w)\}$ for $w \geq a$
and $n = 1, 2, \ldots, k$,

then P_2 ,

Theorem 3.6 .

Suppose that k is any positive non-integral number.

If (i) $\gamma(w)$ is positive and absolutely continuous in every interval $[a, W]$ and $\gamma(w) = O(w)$ for $w \geq a$,

(ii) (a) $\psi(w) = O\left(\{\gamma(w) / w\}^k\right)$ for $w \geq a$,

(b) $w^n \psi^{(n)}(w) = O\left(\{\gamma(w) / w\}^{p+1-n}\right)$ for $w \geq a$
and $n = 1, 2, \ldots, p+1$,

(iii) (a) $\{\gamma(w)\}^n \phi^{(n)}(w) = O\{\phi(w)\}$ for $w \geq a$
and $n = 1, 2, \ldots, p+1$,

(b) $\left\{\dfrac{\phi(w)}{\phi'(w)}\right\}^{n-1} \dfrac{\phi^{(n)}(w)}{\phi'(w)}$ is of bounded variation with respect to w in the range $[a, \infty)$
for $n = 1, 2, \ldots, p+1$,

(iv) $\left\{\frac{w\,\phi'(w)}{\phi(w)}\right\}^k \psi(w)$ is of bounded variation with respect to w in the range $[a,\infty)$,

(v) uniformly in $0 < v < 1$ and for $t \geq a$,

(a) $\left\{\frac{x\,(1-v)\,\phi'(t+vx)}{\phi(t+x) - \phi(t+vx)}\right\}^{p+1-k}$ is of bounded variation with respect to x in the range $[0,\infty)$ †,

(b) $\left\{\frac{\phi(t+vx)}{\phi(t+x)}\right\}^{k-p}$ is of bounded variation with respect to x in the range $[0,\infty)$,

then P_2 ,

Theorem 3.7 .

Suppose that k is any positive non-integral number. If conditions (i) to (iv) inclusive of theorem 3.6 hold, and, in addition,

(v)' (a) $\frac{w\,\phi''(w)}{\phi'(w)}$ is non-negative monotonic non-decreasing for $w \geq a$,

(b) $\frac{w\,\phi'(w)}{\phi(w)}$ is non-negative monotonic non-decreasing for $w \geq a$,

then P_2 ,

---

† The value of the expression at $x = 0$ is defined to be

$$\lim_{x\to 0+} \left\{\frac{x\,(1-v)\,\phi'(t+vx)}{\phi(t+x) - \phi(t+vx)}\right\}^{p+1-k} = 1 .$$

Theorem 3.8 .

Suppose that k *is any positive non-integral number.*

If (i) $\phi(w)$ *is a logarithmico-exponential function* ,

$$\text{(ii)} \quad \frac{1}{w} \prec \frac{\phi'(w)}{\phi(w)} \prec 1 ,$$

$$\text{(iii)} \quad \psi(w) = \left\{\frac{\phi(w)}{w\,\phi'(w)}\right\}^{k} ,$$

then P_2 ,

and

Theorem 3.9 .

Suppose that k *is any positive non-integral number.*

If (i) $\phi(w)$ *is a logarithmico-exponential function* ,

$$\text{(ii)} \quad \frac{1}{w} \preccurlyeq \frac{\phi'(w)}{\phi(w)} ,$$

$$\text{(iii)} \quad \psi(w) = \left\{\frac{\phi(w)}{w\,\phi'(w)}\right\}^{k} ,$$

then P_2 .

For integral values of k the hypotheses of theorem 3.5 are satisfied when $\phi(w)$, $\psi(w)$ are as in theorem 3.4 and $\gamma(w) = \phi(w) / \phi'(w)$. Theorem 3.2 is included in theorem 3.5, the only difference in the hypotheses being that where Dikshit has

$\gamma(w) / w$ <u>is of bounded variation for</u> $w \geq a$,

we have

$\gamma(w) = O(w)$ <u>for</u> $w \geq a$.

Theorems 3.8 and 3.9 are the extensions of theorems 3.3 and 3.4 respectively to non-integral orders of summability.

I have now discovered that, although Dikshit [7] has stated theorem 3.2, he has, in fact, proved theorem 3.5. He has also proved the following theorem [8] :-

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[7] <u>G.D.Dikshit</u>, "On the absolute Riesz summability factors of infinite series, I", <u>Indian Journal Math.</u>, 1, (1958), 33 - 40.

[8] <u>G.D.Dikshit</u>, "On the absolute Riesz summability factors of infinite series, II", <u>Proc. Nat. Inst. Sci. India</u>, 26, A, (1960), 86 - 94.

Theorem 3.10 .

Suppose that k is any positive non-integral number.

If (i) $\gamma(w)$ is positive and absolutely continuous in every interval $[a, W]$ and $\gamma(w) = O(w)$ for $w \geq a$,

(ii) $w^n \psi^{(n)}(w) = O\left(\{\gamma(w) / w\}^{k-n}\right)$ for $w \geq a$ and $n = 0, 1, \ldots, p$,

(iii) $\gamma(w)\, \phi'(w) = O\{\phi(w)\}$ for $w \geq a$ or $\{\gamma(w)\}^n \phi^{(n)}(w) = O\{\phi(w)\}$ for $w \geq a$ and $n = 1, 2, \ldots, p$, according as $0 < k < 1$ or $k > 1$,

(iv) uniformly in $w \geq a$, $w^k \; {}_wD_t^k\left(\left\{1 - \frac{\phi(t)}{\phi(w)}\right\}^k \psi(t)\right)$ is of bounded variation with respect to w in the range $[t, \infty)$,

then P_2 .

3.2 Lemmas.

The following lemmas are required :-

Lemma 3.1 .

The n^{th} derivative of $\{f(t)\}^m$ is a sum of a number of terms like

$$M_1 \ \{f(t)\}^{m-\sigma} \ \prod_{\nu=1}^{n} \left\{f^{(\nu)}(t)\right\}^{\alpha_\nu}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ *are non-negative integers such that*

$$1 \leq \sum_{\nu=1}^{n} \alpha_\nu = \sigma \leq \sum_{\nu=1}^{n} \nu\alpha_\nu = n .$$

If m *is a positive integer, then* $\sigma \leq m$.

This is a re-statement of lemma 2.1 .

Lemma 3.2 .

$${}_wI^k \ {}_wD_t^k \left(\left\{1 - \frac{\phi(t)}{\phi(w)}\right\}^k \psi(t)\right) = \left\{1 - \frac{\phi(t)}{\phi(w)}\right\}^k \psi(t) .$$

This is a re-statement of theorem 1.5 .

Lemma 3.3 .

$$\int_0^w \left\{1 - \frac{\phi(t)}{\phi(w)}\right\}^k \psi(t) \quad dA(t)$$

$$= \frac{1}{\Gamma(k+1)} \int_0^w {}_wD_t^k\left(\left\{1 - \frac{\phi(t)}{\phi(w)}\right\}^k \psi(t)\right) \left\{\frac{d}{dt} A_k(t)\right\} \quad dt .$$

This is a re-statement of theorem 1.6 .

Lemma 3.4 .

Assuming that the expressions below have a meaning,

(i) if $$G_1(w) = \int_a^w f_1(w,t)\, g_1(t)\ dt$$

then

$$\int_a^\infty |dG_1(w)| \leq \overline{\mathrm{bd}}_{a<t} \left\{ |f_1(t,t)| + \int_t^\infty |d_w f_1(w,t)| \right\} \cdot \int_a^\infty |g_1(t)|\ dt ,$$

(ii) if $$G_2(w) = \int_{a/w}^1 f_2(w,t)\, g_2(t)\ dt$$

then

$$\int_a^\infty |dG_2(w)| \leq \overline{\mathrm{bd}}_{0<t<1} \left\{ |f_2(\tfrac{a}{t},t)| + \int_{a/t}^\infty |d_w f_2(w,t)| \right\} \cdot \int_0^1 |g_2(t)|\ dt ,$$

(iii) if $$G_3(w) = \int_0^1 f_3(w,t)\, g_3(t)\ dt$$

then

$$\int_a^\infty |dG_3(w)| \leq \overline{\mathrm{bd}}_{0<t<1} \left\{ \int_a^\infty |d_w f_3(w,t)| \right\} \cdot \int_0^1 |g_3(t)|\ dt .$$

Proof.

(i) Express $G_1(w)$ in the form

$$G_1(w) = \int_a^\infty f_1(w,t)\, g_1(t)\, dt$$

where $f_1(w,t) = 0$ when $t > w$, and then, for every finite monotonic non-decreasing sequence $\{w_n\}$, with $w_1 > 0$,

$$\Sigma \left| G_1(w_{n+1}) - G_1(w_n) \right|$$

$$= \Sigma \left| \int_a^\infty \left\{ f_1(w_{n+1},t) - f_1(w_n,t) \right\} g_1(t)\, dt \right|$$

$$\leq \Sigma \int_a^\infty \left| f_1(w_{n+1},t) - f_1(w_n,t) \right| . \left| g_1(t) \right| dt$$

$$= \int_a^\infty \left\{ \Sigma \left| f_1(w_{n+1},t) - f_1(w_n,t) \right| \right\} . \left| g_1(t) \right| dt$$

$$\leq \overline{\mathrm{bd}}_{a<t} \left\{ \int_a^\infty \left| d_w f_1(w,t) \right| \right\} . \int_a^\infty \left| g_1(t) \right| dt .$$

Hence

$$\int_a^\infty \left| dG_1(w) \right| \leq \overline{\mathrm{bd}}_{a<t} \left\{ \int_a^\infty \left| d_w f_1(w,t) \right| \right\} . \int_a^\infty \left| g_1(t) \right| dt$$

$$= \overline{\mathrm{bd}}_{a<t} \left\{ \left| f_1(t,t) \right| + \int_t^\infty \left| d_w f_1(w,t) \right| \right\} .$$

$$: \int_a^\infty \left| g_1(t) \right| dt .$$

Similar methods can be used to prove parts (ii) and (iii). This lemma is due to Tatchell : see [21], lemma 1.

Lemma 3.5.

Uniformly for $\sigma = 0, 1, \ldots, p$ and $t \geq a$,

$$\Phi_\sigma(w,t) = \{\phi(w)\}^{-k} \{\phi(w) - \phi(t)\}^{k-\sigma} \{\phi(t)\}^{\sigma}$$

is of bounded variation with respect to w in the range $[t,\infty)$.

Proof.

(i) $\sigma = 0$.

$$\Phi_0(w,t) = \left\{1 - \frac{\phi(t)}{\phi(w)}\right\}^k$$

is clearly of bounded variation with respect to w in the range $[t,\infty)$, uniformly for $t \geq a$.

(ii) $\sigma > 0$.

$$\frac{\frac{\partial}{\partial w}\Phi_\sigma(w,t)}{\Phi_\sigma(w,t)} = \frac{-k\,\phi'(w)}{\phi(w)} + \frac{(k-\sigma)\,\phi'(w)}{\phi(w) - \phi(t)}$$

$$= \frac{\phi'(w)}{\phi(w)} \left\{\frac{k\,\phi(t) - \sigma\,\phi(w)}{\phi(w) - \phi(t)}\right\}.$$

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[21] J.B.Tatchell, "A theorem on absolute Riesz summability", Journal London Math. Soc., 29, (1954), 49 - 59.

Hence $\Phi_\sigma(w,t)$ is positive monotonic non-decreasing until $k\,\phi(t) = \sigma\,\phi(w)$, and is then monotonic non-increasing as w tends to infinity. Hence the total variation of $\Phi_\sigma(w,t)$ is at most

$$2\,\{\phi(w)\}^{-k}\,\{\phi(w)\}^{k-\sigma}\left(1-\frac{\sigma}{k}\right)^{k-\sigma}\{\phi(w)\}^{\sigma}\left(\frac{\sigma}{k}\right)^{\sigma}$$

which is equal to

$$2\,(k-\sigma)^{k-\sigma}\,k^{-k}\,\sigma^{\sigma}$$

which is constant. Hence $\Phi_\sigma(w,t)$ is of bounded variation with respect to w in the range $[t,\infty)$ uniformly for $t \geq a$.

Lemma 3.6 .

(i) If $u\,\phi''(u)\,/\,\phi'(u)$ is non-negative non-decreasing for $u \geq a$, then

$$F_1(x) = \frac{x\,(1-v)\,\phi'(t+vx)}{\phi(t+x) - \phi(t+vx)}$$ is a [redacted]

Monotonic non-increasing function of x in the range $[0,\infty)$ for $0 < v < 1$ and $t \geq a$.

(ii) If $u\,\phi'(u)\,/\,\phi(u)$ is non-negative non-decreasing for $u \geq a$, then

$$F_2(x) = \frac{\phi(t+vx)}{\phi(t+x)}$$ is a [redacted]

monotonic non-increasing function of x in the range $[0,\infty)$ for $0 < v < 1$ and $t \geq a$,

Proof.

$$(1)\quad \frac{F_1'(x)}{F_1(x)} = \frac{v\,\phi''(t+vx)}{\phi'(t+vx)} + \frac{1}{x} - \frac{\phi'(t+x) - v\,\phi'(t+vx)}{\phi(t+x) - \phi(t+vx)}$$

$$= \frac{v\,\phi''(t+vx)}{\phi'(t+vx)} + \frac{\phi(t+x) - x\,\phi'(t+x)}{x\,\{\phi(t+x) - \phi(t+vx)\}}$$

$$- \frac{\phi(t+vx) - vx\,\phi'(t+vx)}{x\,\{\phi(t+x) - \phi(t+vx)\}}$$

$$= \frac{v\,\phi''(t+vx)}{\phi'(t+vx)} + \frac{F_3(x) - F_3(vx)}{x\,\{\phi(t+x) - \phi(t+vx)\}}$$

where $F_3(x) = \phi(t+x) - x\,\phi'(t+x)$

and so $F_3'(x) = -x\,\phi''(t+x)$.

Hence

$$\frac{F_1'(x)}{F_1(x)} = \frac{v\,\phi''(t+vx)}{\phi'(t+vx)} + \frac{F_3'(\theta x)}{x\,\phi'(t+\theta x)} \qquad \text{where } v < \theta < 1 ,$$

$$= \frac{v\,\phi''(t+vx)}{\phi'(t+vx)} - \frac{\theta\,\phi''(t+\theta x)}{\phi'(t+\theta x)}$$

$$= \frac{1}{x}\left\{\frac{(t+vx)\,\phi''(t+vx)}{\phi'(t+vx)}\cdot\frac{vx}{t+vx} - \frac{(t+\theta x)\,\phi''(t+\theta x)}{\phi'(t+\theta x)}\cdot\frac{\theta x}{t+\theta x}\right\}$$

$$\leq 0 ,$$

since $u\,\phi''(u) / \phi'(u)$ and $u/(t+u)$ are non-negative non-decreasing functions of u for $u \geq a$. Since $F_1(x)$ is a positive function, it is a monotonic non-increasing function of x for $0 < v < 1$ and $t \geq a$.

(ii)

$$\frac{F_2'(x)}{F_2(x)} = \frac{v\,\phi'(t+vx)}{\phi(t+vx)} - \frac{\phi'(t+x)}{\phi(t+x)}$$

$$= \frac{1}{x}\left\{\frac{(t+vx)\,\phi'(t+vx)}{\phi(t+vx)} \cdot \frac{vx}{t+vx} - \frac{(t+x)\,\phi'(t+x)}{\phi(t+x)} \cdot \frac{x}{t+x}\right\}$$

$$\leq 0 ,$$

since $u\,\phi'(u) / \phi(u)$ and $u/(t+u)$ are non-negative non-decreasing functions of u for $u \geq a$. Since $F_2(x)$ is a positive function, it is a monotonic non-increasing function of x for $0 < v < 1$ and $t \geq a$.

<u>Lemma 3.7 .</u>

<u>If</u> (i) $\phi(w)$ <u>is a logarithmico-exponential function</u> ,

(ii) $\phi(w) = O(w^{\delta})$, $(\delta > 0)$,

<u>then</u> $\sum_{n=1}^{\infty} a_n$ <u>is summable</u> $|R, \phi(\lambda), k|$ <u>whenever</u> $\sum_{n=1}^{\infty} a_n$ <u>is summable</u> $|R, \lambda, k|$.

This is a re-statement of theorem 1.7.2 .

Lemma 3.8 .

If $\phi_1(t)$ is a logarithmico-exponential function and $\phi_1(t) \succ e^t$, then there exists a positive integer r *such that*

$$e_r(t) \prec \phi_1(t) \preccurlyeq e_{r+1}(t)$$

where $e_r(t) = \exp\{e_{r-1}(t)\}$ *and* $e_0(t) = t$.

This is property (iii) of logarithmico-exponential functions as stated in section 1.2 .

Lemma 3.9 .

If $\phi_1(t)$, $\phi_2(t)$ are logarithmico-exponential functions not tending to finite limits, and $\phi_1(t) \preccurlyeq \phi_2(t)$, then

$$\phi_1'(t) \preccurlyeq \phi_2'(t) .$$

This is property (iv) of logarithmico-exponential functions as stated in section 1.2 .

3.3 Proof of theorem 3.5 .

Suppose that k is a positive integer throughout this section. We wish to show that $w^{-k} F_k(w)$ is of bounded variation with respect to w in the range $[0,\infty)$ whenever $w^{-k} A_k(w)$ is of bounded variation with respect to w in the range $[0,\infty)$. Without loss of generality, we assume that

$$A(w) = 0 \quad \text{for} \quad 0 \leq w \leq a \ ,$$

and note that it is sufficient to prove that $\{\phi(w)\}^{-k} G_k(w)$ is of bounded variation with respect to w in the range $[a,\infty)$.

Now, since $w^{-k} A_k(w)$ is of bounded variation with respect to w in the range $[a,\infty)$, we have that

$$\int_a^w \left| \frac{d}{dt} \{t^{-k} A_k(t)\} \right| dt = O(1) \ . \tag{3.3.1}$$

It is clear that we can express $\{\phi(w)\}^{-k} G_k(w)$ in the form

$$\{\phi(w)\}^{-k} G_k(w) = \{\phi(w)\}^{-k} \int_a^w \{\phi(w) - \phi(t)\}^k \psi(t) \ dA(t) \ ,$$

$$= \frac{(-1)^k}{\Gamma(k+1)} \int_a^w \left(\frac{\partial}{\partial t}\right)^k \left(\left\{1 - \frac{\phi(t)}{\phi(w)}\right\}^k \psi(t)\right) .$$

$$\cdot \frac{d}{dt}\left\{t^k \frac{A_k(t)}{t^k}\right\} \quad dt ,$$

$$= \frac{1}{\Gamma(k)} I_1 + \frac{1}{\Gamma(k+1)} I_2 ,$$

Where

$$I_1 = \int_a^w \left(\frac{\partial}{\partial t}\right)^k \left(\left\{1 - \frac{\phi(t)}{\phi(w)}\right\}^k \psi(t)\right) \quad t^{k-1} \left\{\frac{A_k(t)}{t^k}\right\} \quad dt ,$$

$$= \int_a^w q_1(w,y) \frac{d}{dy}\left\{\frac{A_k(y)}{y^k}\right\} \quad dy ,$$

where

$$q_1(w,y) = \int_y^w \left(\frac{\partial}{\partial t}\right)^k \left(\left\{1 - \frac{\phi(t)}{\phi(w)}\right\}^k \psi(t)\right) \quad t^{k-1} \quad dt ,$$

and

$$I_2 = \int_a^w \left(\frac{\partial}{\partial t}\right)^k \left(\left\{1 - \frac{\phi(t)}{\phi(w)}\right\}^k \psi(t)\right) \quad t^k \frac{d}{dt}\left\{\frac{A_k(t)}{t^k}\right\} \quad dt .$$

Consider first I_1 . Because of (3.3.1) and since $q_1(y,y) = 0$, in order to show that I_1 is of bounded variation with respect to w in the range $[a,\infty)$, it is sufficient, in view of lemma 3.4, to show that

$$\int_y^\infty |d_w q_1(w,y)| < M \quad \text{for} \quad y \geqq a .$$

Integrating $q_1(w,y)$ out by parts, we obtain

$$q_1(w,y) = b_0 \, \psi(y) \, \Phi_0(w,y)$$

$$+ \sum_{r=1}^{k-1} b_r \, y^r \left(\frac{\partial}{\partial y}\right)^r \left(\left\{1 - \frac{\phi(y)}{\phi(w)}\right\}^k \psi(y)\right) ,$$

where $b_0, b_1, \ldots, b_{k-1}$ are constants and

$$\Phi_0(w,y) = \left\{1 - \frac{\phi(y)}{\phi(w)}\right\}^k .$$

Now, in view of lemma 3.5, and since, by conditions (i) and (ii) of theorem 3.5, $\psi(y) = O(1)$ for $y \geqq a$,

$$\psi(y) \;\; \Phi_0(w,y)$$

is of bounded variation with respect to w in the range $[y,\infty)$ uniformly for $y \geqq a$.

Also, by lemma 3.1 and Leibnitz's theorem on the differentiation of a product,

$$y^r \left(\frac{\partial}{\partial y}\right)^r \left(\left\{1 - \frac{\phi(y)}{\phi(w)}\right\}^k \psi(y)\right)$$

can be expressed as a sum of constant multiples of terms like

$$q_2(w,y) = y^r \psi^{(r-n)}(y) \{\phi(y)\}^{-\sigma} \prod_{\nu=1}^{n} \{\phi^{(\nu)}(y)\}^{\alpha_\nu} .$$

$$\cdot \ \Phi_\sigma(w,y) ,$$

where $\alpha_1 , \alpha_2 , \ldots , \alpha_n$ are non-negative integers such that

$$1 \leq \sum_{\nu=1}^{n} \alpha_\nu = \sigma \leq \sum_{\nu=1}^{n} \nu\alpha_\nu = n \leq r \leq k-1 ,$$

and

$$\Phi_\sigma(w,y) = \{\phi(w)\}^{-k} \{\phi(w) - \phi(y)\}^{k-\sigma} \{\phi(y)\}^{\sigma} .$$

Now, since, by conditions (i) , (ii) and (iii) of theorem 3.5

$$y^r \psi^{(r-n)}(y) \{\phi(y)\}^{-\sigma} \prod_{\nu=1}^{n} \{\phi^{(\nu)}(y)\}^{\alpha_\nu}$$

$$= O\left(y^r \{\gamma(y)\}^{k-r+n} y^{-k} \{\phi(y)\}^{-\sigma} \{\gamma(y)\}^{-n} \{\phi(y)\}^{\sigma}\right) ,$$

$$= O\left(\{\gamma(y) / y\}^{k-r}\right) ,$$

$$= O(1) , \text{ since } k-r > 0 ,$$

in view of lemma 3.5 , $q_2(w,y)$ is of bounded variation with respect to w in the range $[y,\infty)$ uniformly for $y \geq a$. Hence I_1 is of bounded variation with respect to w in the range $[a,\infty)$.

Consider now I_2 . Let

$$q_3(w,t) = t^k \left(\frac{\partial}{\partial t}\right)^k \left(\left\{1 - \frac{\phi(t)}{\phi(w)}\right\}^k \psi(t)\right) .$$

In view of (3.3.1) and since, by conditions (ii) and (iii) of theorem 3.5

$$q_3(t,t) = (-1)^k k! \quad t^k \{\phi'(t)\}^k \{\phi(t)\}^{-k} \psi(t)$$

$$= O(1) ,$$

in order to show that I_2 is of bounded variation with respect to w in the range $[a,\infty)$, it is sufficient to show that

$$\int_t^\infty |d_w q_3(w,t)| < M \quad \text{for} \quad t \geq a .$$

By lemma 3.1 and Leibnitz's theorem on the differentiation of a product, $q_3(w,t)$ can be expressed as a sum of constant multiples of terms like

$$q_4(w,t) = t^k \psi^{(k-n)}(t) \{\phi(t)\}^{-\sigma} \prod_{\nu=1}^{n} \{\phi^{(\nu)}(t)\}^{\beta_\nu} .$$

$$. \Phi_\sigma(w,t)$$

where $\beta_1 , \beta_2 , \ldots , \beta_n$ are non-negative integers such that

$$1 \leq \sum_{\nu=1}^{n} \beta_\nu = \sigma \leq \sum_{\nu=1}^{n} \nu\beta_\nu = n \leq k ,$$

and

$$\Phi_\sigma(w,t) = \{\phi(w)\}^{-k} \{\phi(w) - \phi(t)\}^{k-\sigma} \{\phi(t)\}^{\sigma} .$$

Now, since, by conditions (ii) and (iii) of theorem 3.5

$$t^k \psi^{(k-n)}(t) \{\phi(t)\}^{-\sigma} \prod_{\nu=1}^{n} \{\phi^{(\nu)}(t)\}^{\beta_\nu}$$

$$= O\left(t^k \{\gamma(t)\}^n t^{-k} \{\phi(t)\}^{-\sigma} \{\gamma(t)\}^{-n} \{\phi(t)\}^{\sigma} \right) ,$$

$$= O(1) ,$$

in view of lemma 3.5 , $q_4(w,t)$ is of bounded variation with respect to w in the range $[t,\infty)$ uniformly for $t \geq a$.

Hence I_2 is of bounded variation with respect to w in the range $[a,\infty)$.

Hence $\{\phi(w)\}^{-k}\, G_k(w)$ is of bounded variation with respect to w in the range $[a,\infty)$.

This completes the proof of theorem 3.5 .

3.4 Proof of theorem 3.6 .

Suppose throughout this section that k is any positive non-integral number. We wish to show that $w^{-k} F_k(w)$ is of bounded variation with respect to w in the range $[0,\infty)$ whenever $w^{-k} A_k(w)$ is of bounded variation with respect to w in the range $[0,\infty)$. Without loss of generality, we assume that

$$A(w) = 0 \qquad \text{for} \qquad 0 \leqq w \leqq a \ ,$$

and note that it is sufficient to prove that $\{\phi(w)\}^{-k} G_k(w)$ is of bounded variation with respect to w in the range $[a,\infty)$.

Now, since $w^{-k} A_k(w)$ is of bounded variation with respect to w in the range $[a,\infty)$, we have that

$$\int_a^w \left|\frac{d}{dt} \{t^{-k} A_k(t)\}\right| \ dt = O(1) \ . \tag{3.4.1}$$

In view of lemma 3.3 , it is evident that we can express $\{\phi(w)\}^{-k} G_k(w)$ in the form

$$\{\phi(w)\}^{-k} G_k(w) = \{\phi(w)\}^{-k} \int_a^w \{\phi(w) - \phi(t)\}^k \psi(t) \ dA(t) \ ,$$

$$= \frac{1}{\Gamma(k+1)} \int_a^w {}_wD_t^k \left(\left\{ 1 - \frac{\phi(t)}{\phi(w)} \right\}^k \psi(t) \right) .$$

$$. \frac{d}{dt} \left\{ t^k \; \frac{A_k(t)}{t^k} \right\} \; dt \; ,$$

$$= \frac{1}{\Gamma(k)} \; I_1 \; + \; \frac{1}{\Gamma(k+1)} \; I_2$$

where

$$I_1 = \int_a^w {}_wD_t^k \left(\left\{ 1 - \frac{\phi(t)}{\phi(w)} \right\}^k \psi(t) \right) \; t^{k-1} \left\{ \frac{A_k(t)}{t^k} \right\} \; dt \; ,$$

$$= \int_a^w q_1(w,y) \; \frac{d}{dy} \left\{ \frac{A_k(y)}{y^k} \right\} \; dy \; ,$$

where

$$q_1(w,y) = \int_y^w {}_wD_t^k \left(\left\{ 1 - \frac{\phi(t)}{\phi(w)} \right\}^k \psi(t) \right) \; t^{k-1} \; dt \; ,$$

and

$$I_2 = \int_a^w {}_wD_t^k \left(\left\{ 1 - \frac{\phi(t)}{\phi(w)} \right\}^k \psi(t) \right) \; t^k \; \frac{d}{dt} \left\{ \frac{A_k(t)}{t^k} \right\} \; dt \; .$$

Consider first I_1 . Because of (3.4.1) and since $q_1(y,y) = 0$, in order to show that I_1 is of bounded variation with respect to w in the range $[a,\infty)$, it is sufficient, in view of lemma 3.4 , to show that

$$\int_{y}^{\infty} \| d_w q_1(w,y) \| < M \quad \text{for} \quad y \geq a \ .$$

Now, in view of lemma 3.2 , (see (1.3))

$$(-1)^{p+1} \Gamma(p+1-k) \, q_1(w,y)$$

$$= \int_{y}^{w} t^{k-1} \, dt \int_{t}^{w} (u-t)^{p-k} \left(\frac{\partial}{\partial u}\right)^{p+1} \left(\left\{1 - \frac{\phi(u)}{\phi(w)}\right\}^{k} \psi(u) \right) du \ ,$$

$$= \int_{y}^{w} \left(\frac{\partial}{\partial u}\right)^{p+1} \left(\left\{1 - \frac{\phi(u)}{\phi(w)}\right\}^{k} \psi(u) \right) du \int_{y}^{u} (u-t)^{p-k} \, t^{k-1} \, dt \ ,$$

$$= \int_{y}^{w} u^{p} \left(\frac{\partial}{\partial u}\right)^{p+1} \left(\left\{1 - \frac{\phi(u)}{\phi(w)}\right\}^{k} \psi(u) \right) du \int_{y/u}^{1} (1-v)^{p-k} \, v^{k-1} \, dv \ ,$$

$$= \int_{y/w}^{1} (1-v)^{p-k} \, v^{k-1} \, dv \int_{y/v}^{w} u^{p} \left(\frac{\partial}{\partial u}\right)^{p+1} \left(\left\{1 - \frac{\phi(u)}{\phi(w)}\right\}^{k} \psi(u) \right) du \ ,$$

$$= \int_{y/w}^{1} (1-v)^{p-k} \, v^{k-1} \, q_2(w, y/v) \, dv \ ,$$

where

$$q_2(w,z) = \int_{z}^{w} u^{p} \left(\frac{\partial}{\partial u}\right)^{p+1} \left(\left\{1 - \frac{\phi(u)}{\phi(w)}\right\}^{k} \psi(u) \right) du \ ,$$

and $z = y/v$.

Again, in view of lemma 3.4 , since

$$\int_0^1 (1-v)^{p-k}\, v^{k-1}\, dv$$

is finite, in order to show that I_1 is of bounded variation with respect to w in the range $[a,\infty)$, it is sufficient to show that

$$\int_z^\infty |d_w q_2(w,z)| < M \qquad \text{for} \qquad z \geq a \;.$$

Integrating $q_2(w,z)$ out by parts, we obtain

$$q_2(w,z) = c_0\, \psi(z)\, \Phi_0(w,z)$$

$$+ \sum_{r=1}^{p} c_r\, z^r \left(\frac{\partial}{\partial z}\right)^r \left(\left\{1 - \frac{\phi(z)}{\phi(w)}\right\}^k \psi(z)\right)$$

where c_0 , c_1 , ... , c_p are constants and

$$\Phi_0(w,z) = \left\{1 - \frac{\phi(z)}{\phi(w)}\right\}^k \;.$$

Now, in view of lemma 3.5 , and since, by conditions (i) and (ii) of theorem 3.6 , $\psi(z) = O(1)$ for $z \geq a$,

$$\psi(z)\, \Phi_0(w,z)$$

is of bounded variation with respect to w in the range $[z,\infty)$ uniformly for $z \geq a$.

Also, by lemma 3.1 and Leibnitz's theorem on the differentiation of a product

$$z^r \left(\frac{\partial}{\partial z}\right)^r \left(\left\{1 - \frac{\phi(z)}{\phi(w)}\right\}^k \psi(z)\right)$$

can be expressed as a sum of constant multiples of terms like

$$q_3(w,z) = z^r \psi^{(r-n)}(z) \{\phi(z)\}^{-\sigma} \prod_{\nu=1}^{n} \left\{\phi^{(\nu)}(z)\right\}^{\alpha_\nu} .$$

$$. \Phi_\sigma(w,z)$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are non-negative integers such that

$$1 \leq \sum_{\nu=1}^{n} \alpha_\nu = \sigma \leq \sum_{\nu=1}^{n} \nu\alpha_\nu = n \leq r \leq p ,$$

and

$$\Phi_\sigma(w,z) = \{\phi(w)\}^{-k} \{\phi(w) - \phi(z)\}^{k-\sigma} \{\phi(z)\}^{\sigma} .$$

Now, in view of lemma 3.5, $q_3(w,z)$ is of bounded variation with respect to w in the range $[z,\infty)$ uniformly for $z \geq a$, since, by conditions (ii) and (iii) of theorem 3.6

$$z^r\ \psi^{(r-n)}(z)\ \{\phi(z)\}^{-\sigma}\ \prod_{\nu=1}^{n}\{\phi^{(\nu)}(z)\}^{\alpha_\nu}$$

$$= O\left(z^r\ \{\gamma(z)\}^{m-r+n}\ z^{-m}\ \{\phi(z)\}^{-\sigma}\ \{\gamma(z)\}^{-n}\ \{\phi(z)\}^{\sigma}\right) ,$$

$$= O\left(\{\gamma(z)\ /\ z\}^{m-r}\right) ,$$

$$= O(1) \qquad \text{since} \qquad m-r > 0 \ , \quad \text{where}$$

$$m = \begin{cases} p+1 & \text{if} \quad r > n \ , \\ k & \text{if} \quad r = n \ . \end{cases}$$

Hence I_1 is of bounded variation with respect to w in the range $[a,\infty)$.

Consider now I_2 . In order to show that I_2 is of bounded variation with respect to w in the range $[a,\infty)$, because of (3.4.1) and in view of lemmas 3.4 and 3.2 (1.3) it is sufficient to show that

$$Q(w,t) = t^k \int_t^w (u-t)^{p-k} \left(\frac{\partial}{\partial u}\right)^{p+1} \left(\left\{1 - \frac{\phi(u)}{\phi(w)}\right\}^k \psi(u)\right) du$$

is of bounded variation with respect to w in the range $[t,\infty)$ uniformly for $t \geq a$, and that $Q(t,t)$, defined to be $\lim\limits_{w \to t+} Q(w,t)$, is bounded.

Now, for $t < u < w$

$$t^k (u-t)^{p-k} \left(\frac{\partial}{\partial u}\right)^{p+1} \left(\left\{ 1 - \frac{\phi(u)}{\phi(w)} \right\}^k \psi(u) \right)$$

$$= k(k-1) \ldots (k-p) \; t^k (u-t)^{p-k} (w-u)^{k-p-1} \;.$$

$$\cdot \left(\psi(w) \left\{ \frac{\phi'(w)}{\phi(w)} \right\}^k + \delta(w,u) \right)$$

where $\delta(w,u) \to 0$ as $w \to t+$ uniformly for $w > u > t$.

Hence

$$Q(t,t) = \Gamma(p+1-k)\,\Gamma(k+1) \left\{ \frac{t\,\phi'(t)}{\phi(t)} \right\}^k \psi(t)$$

$$= O(1)$$

in view of condition (iv) of theorem 3.6 .

In view of lemma 2.1 and Leibnitz's theorem on the differentiation of a product, it is sufficient to show that each of the following integrals is of bounded variation with respect to w in the range $[t,\infty)$ uniformly for $t \geqq a$:-

$$I_{21} = \left\{\frac{t}{\phi(w)}\right\}^k \int_t^w (u-t)^{p-k}\, \psi^{(p+1)}(u)\, \{\phi(w) - \phi(u)\}^k \, du ,$$

$$I_{22} = \left\{\frac{t}{\phi(w)}\right\}^k \int_t^w (u-t)^{p-k}\, \psi^{(p+1-r)}(u)\, \{\phi(w) - \phi(u)\}^{k-\sigma} \, .$$

$$. \prod_{\nu=1}^{r} \{\phi^{(\nu)}(u)\}^{\beta_\nu} \, du ,$$

and

$$I_{23} = \left\{\frac{t}{\phi(w)}\right\}^k \int_t^w (u-t)^{p-k}\, \psi(u)\, \{\phi(w) - \phi(u)\}^{k-\mu} \, .$$

$$. \prod_{\nu=1}^{p+1} \{\phi^{(\nu)}(u)\}^{\delta_\nu} \, du ,$$

where

$$1 \leq \sum_{\nu=1}^{r} \beta_\nu = \sigma \leqq \sum_{\nu=1}^{r} \nu\beta_\nu = r \leqq p ,$$

and

$$1 \leqq \sum_{\nu=1}^{p+1} \delta_\nu = \mu \leqq \sum_{\nu=1}^{p+1} \nu\delta_\nu = p+1 .$$

Now, we can write I_{21} in the form

$$I_{21} = \int_t^w \Phi_0(w,u)\ t^k\ (u-t)^{p-k}\ \psi^{(p+1)}(u)\ du$$

where

$$\Phi_0(w,u) = \left\{1 - \frac{\phi(u)}{\phi(w)}\right\}^k .$$

Since, by condition (ii) of theorem 3.6

$$\int_t^\infty \left| t^k\ (u-t)^{p-k}\ \psi^{(p+1)}(u)\right|\ du$$

$$= \int_t^\infty O\left\{t^k\ (u-t)^{p-k}\ u^{-p-1}\right\}\ du$$

and since the latter integral is finite and independent of t , it is clear that, in view of lemmas 3.4 and 3.5 , I_{21} is of bounded variation with respect to w in the range $[t,\infty)$ uniformly for $t \geq a$.

Also, we can write I_{22} in the form

$$I_{22} = \int_t^w \Phi_\sigma(w,u)\ t^k\ (u-t)^{p-k}\ \psi^{(p+1-r)}(u)\ \{\phi(u)\}^{-\sigma} .$$

$$. \prod_{\nu=1}^{r} \left\{\phi^{(\nu)}(u)\right\}^{\beta_\nu}\ du ,$$

where

$$\Phi_\sigma(w,u) = \{\phi(w)\}^{-k}\{\phi(w) - \phi(u)\}^{k-\sigma}\{\phi(u)\}^{\sigma} ,$$

and

$$1 \leq \sum_{\nu=1}^{r} \beta_\nu = \sigma \leq \sum_{\nu=1}^{r} \nu\beta_\nu = r \leq p .$$

Since, by conditions (ii) and (iii) of theorem 3.6

$$\int_t^\infty \left| t^k (u-t)^{p-k} \psi^{(p+1-r)}(u) \{\phi(u)\}^{-\sigma} \prod_{\nu=1}^{r} \left\{\phi^{(\nu)}(u)\right\}^{\beta_\nu} du \right|$$

$$= \int_t^\infty O\left\{t^k (u-t)^{p-k} u^{-p-1}\right\} du ,$$

and, again, since this latter integral is finite and independent of t, it is clear, in view of lemmas 3.4 and 3.5, that I_{22} is of bounded variation with respect to w in the range $[t,\infty)$ uniformly for $t \geq a$.

Finally, we write I_{23} in the form

$$I_{23} = \left\{\frac{t}{\phi(w)}\right\}^k \int_t^w (u-t)^{p-k} \psi(u) \{\phi(w) - \phi(u)\}^{k-p-1} .$$

$$. \{\phi(w) - \phi(u)\}^{p+1-\mu} \prod_{\nu=1}^{p+1} \left\{\phi^{(\nu)}(u)\right\}^{\delta_\nu} du .$$

Since $p+1-\mu$ is a positive integer, or zero, $\{\phi(w) - \phi(u)\}^{p+1-\mu}$ can be expandeddby the binomial theorem, giving

$$I_{23} = \sum_{r=0}^{p+1-\mu} (-1)^r \, {}^{p+1-\mu}C_r \quad t^k \, \{\phi(w)\}^{p+1-k-\mu-r} \quad .$$

$$. \int_t^w (u-t)^{p-k} \, \{\phi(w) - \phi(u)\}^{k-p-1} \, \{\phi(u)\}^r \quad .$$

$$. \quad \psi(u) \quad \prod_{\nu=1}^{p+1} \left\{\phi^{(\nu)}(u)\right\}^{\delta_\nu} \quad du \quad ,$$

where the ${}^{p+1-\mu}C_r$ are the appropriate binomial coefficients

To the typical integral of the above sum, apply the transformation :-

$$\begin{cases} u = t + vx \\ w = t + x \end{cases}$$

to obtain

$$q_4(x,t) = t^k \{\phi(t+x)\}^{p+1-k-\mu-r} \int_0^1 \left\{\frac{-\phi(t+vx) + \phi(t+x)}{x\,(1-v)}\right\}^{k-p-1} \quad .$$

$$. \left\{\phi(t+vx)\right\}^r \, \psi(t+vx) \quad \prod_{\nu=1}^{p+1} \left\{\phi^{(\nu)}(t+vx)\right\}^{\delta_\nu} \quad .$$

$$. \quad v^{p-k} \, (1-v)^{k-p-1} \quad dv \quad ,$$

where

$$1 \leq \sum_{\nu=1}^{p+1} \delta_\nu = \mu \leq \sum_{\nu=1}^{p+1} \nu\delta_\nu = p+1 \ .$$

It is sufficient to show that $q_4(x,t)$ is of bounded variation with respect to x in the range $[0,\infty)$ uniformly for $t \geq a$. Now, we can write $q_4(x,t)$ in the form

$$q_4(x,t) = \int_0^1 v^{p-k} (1-v)^{k-p-1} \left\{\frac{\phi(t+x) - \phi(t+vx)}{x\ (1-v)\ \phi'(t+vx)}\right\}^{k-p-1} \cdot$$

$$\cdot \left\{\frac{\phi(t+vx)}{\phi(t+x)}\right\}^{k-p-1+\mu+r} \left\{\frac{t}{t+vx}\right\}^{k} \cdot$$

$$\cdot \ \psi(t+vx) \left\{\frac{(t+vx)\ \phi'(t+vx)}{\phi(t+vx)}\right\}^{k} \cdot$$

$$\cdot \prod_{\nu=1}^{p+1} \left(\left\{\frac{\phi(t+vx)}{\phi'(t+vx)}\right\}^{\nu-1} \frac{\phi^{(\nu)}(t+vx)}{\phi'(t+vx)}\right)^{\delta_\nu} dv \ .$$

Now

$$\int_0^1 v^{p-k} (1-v)^{k-p-1} \ dv$$

is finite, and each of

$$(3.4.2) \qquad \left\{\frac{x\ (1-v)\ \phi'(t+vx)}{\phi(t+x)\ -\ \phi(t+vx)}\right\}^{p+1-k} ,$$

$$(3.4.3) \qquad \left\{\frac{\phi(t+vx)}{\phi(t+x)}\right\}^{k-p-1+\mu+r} ,$$

$$(3.4.4) \qquad \left\{\frac{t}{t+vx}\right\}^{k} ,$$

$$(3.4.5) \qquad \psi(t+vx) \quad \left\{\frac{(t+vx)\ \phi'(t+vx)}{\phi(t+vx)}\right\}^{k} ,$$

and

$$(3.4.6) \qquad \prod_{\nu=1}^{p+1} \left(\left\{\frac{\phi(t+vx)}{\phi'(t+vx)}\right\}^{\nu-1} \frac{\phi^{(\nu)}(t+vx)}{\phi'(t+vx)} \right)^{\delta_\nu} ,$$

are of bounded variation with respect to x in the range $[0,\infty)$ uniformly for $0 < v < 1$ and $t \geq a$;

(3.4.2) , because of condition (v) (a) of theorem 3.6
[redacted] ;

(3.4.3) , because of condition (v) (b) of theorem 3.6
since [redacted]$-1+\mu+r > 0$;

(3.4.4) , because it is a bounded monotonic non-decreasing function of x in the range $[0,\infty)$ uniformly for $0 < v < 1$ and $t \geq a$;

(3.4.5) , because of condition (iv) of theorem 3.6 ;

and

(3.4.6) , because of condition (iii) (b) of theorem 3.6 .

Hence, in view of lemma 3.4 , I_{23} is of bounded variation with respect to x in the range $[0,\infty)$ uniformly for $0 < v < 1$ and $t \geq a$.

Hence I_2 is of bounded variation with respect to w in the range $[a,\infty)$.

Hence $\{\phi(w)\}^{-k} G_k(w)$ is of bounded variation with respect to w in the range $[a,\infty)$.

This completes the proof of theorem 3.6 .

3.5 Proof of theorem 3.7 .

In view of the results of lemma 3.6 , the proof of theorem 3.7 is immediate .

3.6 Proof of theorem 3.8 .

It can readily be shown that, if the hypotheses of theorem 3.8 are true, then the hypotheses of theorem 3.7 are satisfied with $\gamma(w) = \phi(w) / \phi'(w)$.

3.7 Proof of theorem 3.9 .

Firstly, as Guha [10] has done in the case when k is a positive integer, with the aid of results, already quoted, by Chandrasekharan (theorem 1.7.2) and by Tatchell (theorem 3.1 or 1.3.2) , theorem 3.8 can be immediately extended to cover logarithmico-exponential functions, $\phi(w)$, satisfying the hypothesis

$$\frac{1}{w} \preccurlyeq \frac{\phi'(w)}{\phi(w)} \preccurlyeq 1 .$$

It remains, therefore, to consider the case of a logarithmico-exponential function, $\phi(w)$, satisfying the hypothesis

$$\frac{\phi'(w)}{\phi(w)} \succ 1 ,$$

that is

$$\phi(w) \succ e^{w} .$$

[10] U.C.Guha, "Convergence factors for Riesz summability", Journal London Math. Soc., 31, (1956), 311 - 319 .

Then, in view of lemma 3.8 , we can conclude that there is a positive integer r such that

$$e_r(w) \prec \phi(w) \preccurlyeq e_{r+1}(w) \quad . \tag{3.7.1}$$

Now, suppose that

$$\phi(w) = e_r\{\theta(w)\} \quad . \tag{3.7.2}$$

Then, from (3.7.1) and (3.7.2)

$$w \preccurlyeq \theta(w) \preccurlyeq e^w ,$$

that is

$$\log w \preccurlyeq \log \theta(w) \preccurlyeq w ,$$

and, therefore, by lemma 3.9 ,

$$\frac{1}{w} \preccurlyeq \frac{\theta'(w)}{\theta(w)} \preccurlyeq 1 \quad . \tag{3.7.3}$$

Now, in view of (3.7.2) and (3.7.3) , we can make repeated applications of the extension to theorem 3.8 already stated, and conclude that

$\sum_{n=1}^{\infty} a_n \psi(\lambda_n)$ is summable $|R, \phi(\lambda), k|$ whenever $\sum_{n=1}^{\infty} a_n$ is summable $|R, \lambda, k|$, where

$$\psi(w) = \left\{\frac{\theta(w)}{w\ \theta'(w)}\right\}^k \frac{1}{\left(e_0\{\theta(w)\}\right)^k \left(e_1\{\theta(w)\}\right)^k \cdots\cdots \left(e_{r-1}\{\theta(w)\}\right)^k}$$

$$= \left\{\frac{\phi(w)}{w\ \phi'(w)}\right\}^k ,$$

by (3.7.2) .

This completes the proof of theorem 3.9 .

This argument is due to Guha [10] .

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[10] U.C.Guha, "Convergence factors for Riesz summability", Journal London Math. Soc., 31, (1956), 311 - 319 .

Chapter IV.

Strong Riesz Summability.

4.1 Introduction.

In this chapter, we seek conditions sufficient for the truth of the proposition :-

P_3 : $\sum_{n=1}^{\infty} a_n \psi(\lambda_n)$ *is summable* $[R, \phi(\lambda), k]$ *whenever*

$\sum_{n=1}^{\infty} a_n$ *is summable* $[R, \lambda, k]$.

The following theorems, all due to Srivastava, are known :

Theorem 4.1 .

If $\phi(w) = e^w$ *and* $\psi(w) = w^{-k}$, *then* P_3 .

This is a particular case of a more general theorem : see theorems 1.3.3 or 1.3.4 , or [21] .

Theorem 4.2 .

$\sum_{n=1}^{\infty} a_n$ *is summable* $[R, \lambda, k+\delta]$ *whenever* $\sum_{n=1}^{\infty} a_n$ *is summable* $[R, \lambda, k]$, *for every* $\delta \geq 0$.

This is a re-statement of theorem 1.1.3 : see [20] .

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[21] P. Srivastava, "Theorems on strong Riesz summability", Quarterly Journal Math., (2), 11, (1960), 229 - 240 .

[20] P. Srivastava, "On strong Rieszian summability of infinite series", Proc. Nat. Inst. Sci. India, 23, A, (1957), 58 - 71 .

<u>Theorem 4.3 .</u>

(i) <u>Suppose that</u> k <u>is a positive integer.</u>

<u>If</u> (a) $\psi(w) = 1$,

(b) $\int_a^w t^k \left|\phi^{(k+1)}(t)\right| dt = O\{\phi(w)\}$ <u>for</u> $w \geq a$,

<u>then</u> P_3 .

(ii) <u>Suppose that</u> k <u>is any positive non-integral number greater than</u> 1 .

<u>If</u> (a) $\psi(w) = 1$,

(b) $\int_a^w t^{p+1} \left|\phi^{(p+2)}(t)\right| dt = O\{\phi(w)\}$ <u>for</u> $w \geq a$,

<u>and either</u>

(c.1) $\phi'(w)$ <u>is monotonic non-decreasing for</u> $w \geq a$,

<u>or</u>

(c.2) $\phi'(w)$ <u>is monotonic non-increasing for</u> $w \geq a$ <u>and</u> $w\,\phi''(w) = O\{\phi'(w)\}$ <u>for</u> $w \geq a$,

<u>then</u> P_3 .

See [19] , theorem 1 .

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[19] <u>P.Srivastava</u>, "On the second theorem of consistency for strong Riesz summability", <u>Indian Journal Math.</u>, 1, (1958), 1 - 16 .

This is similar to Hirst's theorem for ordinary Riesz summability [15] which is true, in the non-integral case, for all k. Srivastava, however, gives a counter-example to show that, for $0 < k < 1$, there is a series which is summable $[R, \lambda, k]$, but is not summable $[R, \log \lambda, k]$. See [19], theorem 2.

From this theorem, we can immediately deduce

<u>Corollary 4.3.1.</u>

<u>For all</u> $k \geq 1$,

<u>if</u> (i) $\phi(w)$ <u>is a logarithmico-exponential function</u>

(ii) $\phi(w) = O(w^{\delta})$, <u>where</u> $\delta > 0$,

(iii) $\psi(w) = 1$,

<u>then</u> P_3.

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[15] <u>K.A.Hirst</u>, "On the second theorem of consistency in the theory of summation by typical means", <u>Proc. London Math. Soc.</u>, (2), 33, (1932), 353 - 366.

[19] <u>P.Srivastava</u>, "On the second theorem of consistency for strong Riesz summability", <u>Indian Journal Math.</u>, 1, (1958), 1 - 16.

We prove the following theorems :-

<u>Theorem 4.4 .</u>

<u>For all</u> $k > 0$,

<u>if</u> (i) $\phi(w)$ <u>is a logarithmico-exponential function</u> ,

$$\text{(ii)} \quad \frac{1}{w} \prec \frac{\phi'(w)}{\phi(w)} \prec 1 ,$$

$$\text{(iii)} \quad \psi(w) = \left\{\frac{\phi(w)}{w\,\phi'(w)}\right\}^k ,$$

<u>then</u> P_3 ,

and

<u>Theorem 4.5 .</u>

<u>For all</u> $k \geq 1$,

<u>if</u> (i) $\phi(w)$ <u>is a logarithmico-exponential function</u> ,

$$\text{(ii)} \quad \frac{1}{w} \preccurlyeq \frac{\phi'(w)}{\phi(w)} ,$$

$$\text{(iii)} \quad \psi(w) = \left\{\frac{\phi(w)}{w\,\phi'(w)}\right\}^k ,$$

<u>then</u> P_3 .

It is important to note the different ranges of order in these two theorems.

4.2 Lemmas.

The following lemmas are required :-

Lemma 4.1 .

The n^{th} derivative of $\{f(t)\}^m$ is a sum of a number of terms like

$$M_1 \{f(t)\}^{m-\sigma} \prod_{\nu=1}^{n} \{f^{(\nu)}(t)\}^{\alpha_\nu} ,$$

where $\alpha_1 , \alpha_2 , \ldots , \alpha_n$ are non-negative integers such that

$$1 \leq \sum_{\nu=1}^{n} \alpha_\nu = \sigma \leq \sum_{\nu=1}^{n} \nu\alpha_\nu = n .$$

If m is a positive integer, then $\sigma \leq m$.

This is a re-statement of lemma 2.1 .

Lemma 4.2 .

If $\theta(t) \geq 0$, $m > 0$ and $m-n > 0$, then the two assertions

$$\text{(i)} \quad \int_0^w \theta(t) \, dt = o(w^m)$$

and

$$\text{(ii)} \quad \int_0^w t^{-n} \theta(t) \, dt = o(w^{m-n})$$

are equivalent, it being assumed that both integrals converge at the origin.

<u>Proof.</u>

Assume that (i) holds ; then

$$\int_0^w t^{-n}\,\theta(t)\ dt$$

$$= \left\{\int_0^w \theta(t)\ dt\right\} w^{-n} + n\int_0^w t^{-n-1}\ dt\left\{\int_0^t \theta(u)\ du\right\},$$

$$= o\left\{w^{m-n} + n\int_0^w t^{m-n-1}\ dt\right\},$$

$$= o(w^{m-n}) .$$

The converse follows in a similar manner if the integrand in (ii) is multiplied by t^n .
Compare lemma 2 in [9] .

<u>Lemma 4.3 .</u>

$\sum_{n=1}^{\infty} a_n = s\ [R,\lambda,k]$ <u>if and only if</u> $\sum_{n=1}^{\infty} a_n = s\ (R,\lambda,k)$

<u>and</u> $\displaystyle\int_0^w dx\ \left| x^{-k}\int_0^x (x-t)^{k-1}\ t\ \ dA(t)\right| = o(w)$.

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[9] <u>M.Glatfeld</u>, "On strong Rieszian summability", <u>Proc. Glasgow Math. Assoc.</u>, 3, (1957), 123 - 131 .

Proof.

We have $B_k(x) = \int_0^x (x-t)^k \; t \; dA(t)$: hence

$$\int_0^w \left| x^{-k} B_{k-1}(x) \right| dx$$

$$= \int_0^w dx \left| x^{-k} \int_0^x (x-t)^{k-1} \; t \; dA(t) \right| ,$$

$$= \int_0^w dx \left| x^{-k} \int_0^x (x-t)^{k-1} \{x - (x-t)\} \; dA(t) \right| ,$$

$$= \int_0^w dx \left| x^{-(k-1)} \int_0^x (x-t)^{k-1} \; dA(t) \right.$$

$$\left. - \; x^{-k} \int_0^x (x-t)^k \; dA(t) \right| ,$$

$$= \int_0^w \left| C_{k-1}(x) - C_k(x) \right| dx ,$$

where

$$C_k(x) = x^{-k} A_k(x)$$

$$= x^{-k} \int_0^x (x-t)^k \; dA(t) .$$

Also, we have that

$$\int_0^w \left| C_{k-1}(x) - s \right| \, dx$$

$$\leq \left\{ \int_0^w \left| C_{k-1}(x) - C_k(x) \right| \, dx + \int_0^w \left| C_k(x) - s \right| \, dx \right\} \tag{4.2.1}$$

and

$$\int_0^w \left| C_{k-1}(x) - C_k(x) \right| \, dx$$

$$\leq \left\{ \int_0^w \left| C_{k-1}(x) - s \right| \, dx + \int_0^w \left| C_k(x) - s \right| \, dx \right\} . \tag{4.2.2}$$

Now, if $\sum_{n=1}^{\infty} a_n = s \ (R,\lambda,k)$, $C_k(x) \to s$ as $x \to \infty$,

and this implies that

$$\int_0^w \left| C_k(x) - s \right| \, dx = o(w) .$$

Also

$$\int_0^w \left| C_{k-1}(x) - s \right| \, dx = o(w)$$

if and only if $\sum_{n=1}^{\infty} a_n = s \left[R,\lambda,k\right]$.

Hence, in view of (4.2.1) , if $\sum_{n=1}^{\infty} a_n = s\ (R,\lambda,k)$, and

$\int_0^w \left| x^{-k} B_{k-1}(x) \right| dx = o(w)$, we can deduce that

$\sum_{n=1}^{\infty} a_n = s\ [R,\lambda,k]$.

Conversely, if $\sum_{n=1}^{\infty} a_n = s\ [R,\lambda,k]$, by theorem 1.2.1 ,

$\sum_{n=1}^{\infty} a_n = s\ (R,\lambda,k)$: hence, in view of (4.2.2) , we

can deduce that $\int_0^w \left| x^{-k} B_{k-1}(x) \right| dx = o(w)$.

For a similar result on strong Cesaro summability, see [16] , theorem 3 .

Lemma 4.4 .

Assuming the expressions below have a meaning,

(i) *if* $G_1(w) = \int_a^w f_1(w,t)\, g_1(t)\, dt$,

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[16] J.M.Hyslop, "Note on the strong summability of series", Proc. Glasgow Math. Assoc., 1, (1952-53), 16 - 20 .

$$\text{then} \quad \int_a^w |dG_1(w)| \leq \overline{\text{bd}}_{a<t<w} \left\{ |f_1(t,t)| + \int_t^w |d_w f_1(w,t)| \right\}$$

$$\cdot \int_a^w |g_1(t)| \ dt \ ,$$

(ii) <u>if</u> $G_2(w) = \int_0^1 f_2(w,t)\, g_2(t) \ dt$,

$$\text{then} \quad \int_a^w |dG_2(w)| \leq \overline{\text{bd}}_{0<t<1} \int_a^w |d_w f_2(w,t)| \ .$$

$$\cdot \int_0^1 |g_2(t)| \ dt \ .$$

This result is similar to lemma 3.4 and is proved in a similar fashion. See, also, [19] , lemma 5 .

<u>Lemma 4.5 .</u>

(i) <u>If</u> $t\, \phi''(t) / \phi'(t)$ <u>is non-negative monotonic non-decreasing for</u> $t \geq a$, <u>then</u>

[19] <u>P.Srivastava</u>, "On the second theorem of consistency for strong Riesz summability", <u>Indian Journal Math.</u>, 1, (1958), 1 - 16 .

$$F_1(y) = \frac{y\ (1-v)\ \phi'(u+vy)}{\phi(u+y) - \phi(u+vy)}$$ <u>is a monotonic non-increasing function of</u> y <u>in the range</u> $[0,\infty)$ [redacted] <u>for</u> $0 < v < 1$ <u>and</u> $u \geq a$.

(ii) <u>If</u> $t\ \phi'(t)\ /\ \phi(t)$ <u>is non-negative monotonic non-decreasing for</u> $t \geq a$, <u>then</u>

$$F_2(y) = \frac{\phi(u+vy)}{\phi(u+y)}$$ <u>is a monotonic non-increasing function of</u> y <u>in the range</u> $[0,\infty)$ [redacted] <u>for</u> $0 < v < 1$ <u>and</u> $u \geq a$.

This is a re-statement of lemma 3.6 .

4.3 Proof of theorem 4.4 .

We assume, without loss of generality, that the sum of the series is zero, and that

$$\Lambda(w) = 0 \quad \text{for} \quad 0 \leq w \leq a \ .$$

In view of theorem 1.2.1 , since $\sum_{n=1}^{\infty} a_n$ is summable $[R,\lambda,k]$ we can deduce that $\sum_{n=1}^{\infty} a_n$ is summable (R,λ,k) , and, in view of theorem 2.3 , that $\sum_{n=1}^{\infty} a_n \psi(\lambda_n)$ is summable $(R,\phi(\lambda),k)$. Hence, in view of lemmas 4.2 and 4.3 , it is sufficient for the proof of theorem 4.4 , to show that

$$\int_a^w \phi'(w) \left|M_{k-1}(x)\right| \; dx = o\left(\{\phi(w)\}^{k+1}\right) \ . \tag{4.3.1}$$

Now, in view of lemma 4.2 , since $\sum_{n=1}^{\infty} a_n$ is summable $[R,\lambda,k]$ to sum zero, we have that

$$\int_a^w \left|A_{k-1}(x)\right| \; dx = o(w^k) \ . \tag{4.3.2}$$

Also, from the conditions of theorem 4.4 , we can deduce the following for $w \geq a$:- (see appendix)

$$w^{-k} \{\phi(w)\}^{k-p} \succ 1 \tag{4.3.3}$$

$$\frac{\phi^{(n)}(w)}{\phi'(w)} \left\{\frac{\phi(w)}{\phi'(w)}\right\}^{n-1} \prec 1 \quad \text{for } n = 1, 2, \ldots, p+1 \tag{4.3.4}$$

$$w^{k} \left\{\frac{\phi'(w)}{\phi(w)}\right\}^{k-n} \psi^{(n)}(w) \prec 1 \quad \text{for } n = 0, 1, \ldots, p+1 \tag{4.3.5}$$

(i) Suppose that k is a positive integer. Now

$$\mathcal{M}_{k-1}(x) = \int_a^x \{\phi(x) - \phi(t)\}^{k-1} \phi(t)\ \psi(t)\ dA(t)$$

can be expressed as a sum of constant multiples of

$$A_{k-1}(x)\ \psi(x)\ \phi(x)\ \{\phi'(x)\}^{k-1}$$

and

$$\int_a^x A_{k-1}(t) \left(\frac{\partial}{\partial t}\right)^k \left(\{\phi(x) - \phi(t)\}^{k-1} \phi(t)\ \psi(t)\right)\ dt\ .$$

Now, in view of (4.3.2) and (4.3.3), we have that

$$\int_a^w \phi'(x) \left|A_{k-1}(x)\ \psi(x)\ \phi(x)\ \{\phi'(x)\}^{k-1}\right|\ dx$$

$$= \int_a^w \left|A_{k-1}(x)\right| \left(\{\phi(x)\}^k\ x^{-k}\ \{\phi'(x)\}^{-k}\right) \phi(x)\ \{\phi'(x)\}^k\ dx$$

$$= \int_a^w \left|A_{k-1}(x)\right|\ x^{-k}\ \{\phi(x)\}^{k+1}\ dx$$

$$\leq \; w^{-k} \{\phi(w)\}^{k+1} \int_a^w |A_{k-1}(x)| \; dx$$

$$= \; o\left(\{\phi(w)\}^{k+1}\right) \; . \tag{4.3.6}$$

Also, in view of lemma 4.1 and Leibnitz's theorem on the differentiation of a product

$$\int_a^x A_{k-1}(t) \left(\frac{\partial}{\partial t}\right)^k \left(\{\phi(x) - \phi(t)\}^{k-1} \; \phi(t) \; \psi(t)\right) \; dt$$

can be expressed as a sum of constant multiples of integrals of types

$$I(x) \; = \; \int_a^x A_{k-1}(t) \; \{\phi(x) - \phi(t)\}^{k-1-\mu} \; \phi^{(k-r-m)}(t) \; \psi^{(m)}(t) \; .$$

$$. \; \prod_{\nu=1}^{r} \left\{\phi^{(\nu)}(t)\right\}^{\alpha_\nu} \; dt$$

where $\alpha_1 , \alpha_2 , \ldots , \alpha_r$ are non-negative integers such that

$$\sum_{\nu=1}^{r} \alpha_\nu \; = \; \mu \quad ; \quad \sum_{\nu=1}^{r} \nu\alpha_\nu \; = \; r \quad ;$$

$$0 \leq \mu \leq k-1 \quad ; \quad 0 \leq \mu \leq r \leq k \quad ;$$

and

$$0 \leq m \leq k-r \; .$$

Now, in view of (4.3.3) , (4.3.4) and (4.3.5) ,

$$\int_a^w \phi'(x)\ \left|I(x)\right|\ dx$$

$$\leq M\int_a^w \phi'(x)\ dx\ \int_a^x \left|A_{k-1}(t)\right| \{\phi(x) - \phi(t)\}^{k-1-\mu} \ .$$

$$. \left(\{\phi'(t)\}^{k-r-m}\{\phi(t)\}^{1-k+r+m}\right) \ .$$

$$. \left(\{\phi(t)\}^{k-m}\{\phi'(t)\}^{m-k}\ t^{-k}\right) \ .$$

$$. \left(\{\phi'(t)\}^{r}\{\phi(t)\}^{\mu-r}\right) dt$$

$$\leq M\int_a^w \phi'(x)\ dx\ \int_a^x \left|A_{k-1}(t)\right| \{\phi(x)\}^{k-1-\mu}\ t^{-k}\{\phi(t)\}^{\mu+1}\ dt$$

$$\leq M\int_a^w \phi'(x)\ \{\phi(x)\}^{k-1-\mu}\ x^{-k}\{\phi(x)\}^{\mu+1}\ dx\ \int_a^x \left|A_{k-1}(t)\right|\ dt$$

$$= o\left(\{\phi(w)\}^{k+1}\right) \ . \qquad (4.3.7)$$

Hence, in view of (4.3.6) and (4.3.7) , we can deduce that (4.3.1) is true.

This completes the proof for integer values of k .

(ii) Suppose that k is any positive non-integral number. The condition (4.3.1) that we must prove, can be written as

$$\int_a^w \left| d_x \, \mu_k(x) \right| = o\left(\{\phi(w)\}^{k+1} \right) . \tag{4.3.8}$$

Now, in view of lemma 4.1 and Leibnitz's theorem on the differentiation of a product,

$$\mu_k(x) = \int_a^x \{\phi(x) - \phi(t)\}^k \, \phi(t) \, \psi(t) \, dA(t)$$

$$= \frac{(-1)^{p+1}}{p!} \int_a^x A_p(t) \left(\frac{\partial}{\partial t}\right)^{p+1} \left(\{\phi(x) - \phi(t)\}^k \, \phi(t) \, \psi(t) \right) dt$$

can be expressed as a sum of constant multiples of integrals of the forms

$$J(x) = \int_a^x A_p(t) \, Q(x,t) \, dt$$

where

$$Q(x,t) = Q_{\mu,r,m}(x,t)$$

$$= \{\phi(x) - \phi(t)\}^{k-\mu} \, \phi^{(p+1-r-m)}(t) \, \psi^{(m)}(t) \, . \, \prod_{\nu=1}^{r} \left\{\phi^{(\nu)}(t)\right\}^{\beta_\nu}$$

where $\beta_1, \beta_2, \ldots, \beta_r$ are non-negative integers such that

$$0 \leq \sum_{\nu=1}^{r} \beta_\nu = \mu \leq \sum_{\nu=1}^{r} \nu\beta_\nu = r \leq p+1$$

and

$$0 \leq m \leq p+1-r \ .$$

Now, we have that

$$J(x) = \int_a^x A_p(t)\, Q(x,t)\, dt$$

$$= \frac{1}{\Gamma(p+1-k)} \int_a^x Q(x,t)\, dt \int_a^t (t-u)^{p-k} A_{k-1}(u)\, du$$

$$= \frac{1}{\Gamma(p+1-k)} \int_a^x A_{k-1}(u)\, du \int_u^x Q(x,t)\,(t-u)^{p-k}\, dt$$

$$= \frac{1}{\Gamma(p+1-k)} \int_a^x A_{k-1}(u)\, q_1(x,u)\, du \ ,$$

where

$$q_1(x,u) = \int_u^x Q(x,t)\,(t-u)^{p-k}\, dt \ .$$

We define

$$q_1(u,u) = \lim_{x \to u+} q_1(x,u) \ .$$

Now, for $x > t$

$$Q(x,t) = (x-t)^{k-p-1} \left(\{\phi'(x)\}^k\, \psi(x)\, \phi(x) + \delta(x,t) \right)$$

where $\delta(x,t) \longrightarrow 0$ as $x \longrightarrow t+$, uniformly for $t \geq a$.

Hence

$$q_1(u,u) = \Gamma(k-p)\,\Gamma(p+1-k)\;\phi(u)\,\psi(u)\,\{\phi'(u)\}^k$$

$$= o\left(u^{-k}\,\{\phi(u)\}^k\right) .$$

Therefore

$$\overline{\mathrm{bd}}_{a<u<w}\left|q_1(u,u)\right| = o\left(w^{-k}\,\{\phi(w)\}^k\right) .$$

Hence, in view of lemma 4.4 (i) and (4.3.2) , in order to prove that

$$\int_a^w \left|dJ(x)\right| = o\left(\{\phi(w)\}^k\right) ,$$

it is sufficient to prove that

$$q_2(w) = \overline{\mathrm{bd}}_{a<u<w} \int_u^w \left| d_x \int_u^x Q(x,t)\,(t-u)^{p-k}\,dt \right|$$

$$= o\left(w^{-k}\,\{\phi(w)\}^k\right) .$$

To this last integral, apply the transformation :-

$$\begin{cases} x = u + y \\ t = u + vy \end{cases}$$

Hence, in view of lemma 4.4 (ii), and since

$$\int_0^1 v^{p-k} (1-v)^{k-p-1} \, dv$$

is finite,

$q_2(w)$

$$= \mathcal{O}\left\{ \overline{\mathrm{bd}}_{a<u<w} \int_0^{w-u} \left| d_y \int_0^1 \frac{Q(u+y, u+vy)}{\{y(1-v)\}^{k-p-1}} v^{p-k} (1-v)^{k-p-1} \, dv \right| \right\}$$

$$= \mathcal{O}\left\{ \overline{\mathrm{bd}}_{a<u<w} \; \overline{\mathrm{bd}}_{0<v<1} \int_0^{w-u} \left| d_y \frac{Q(u+y, u+vy)}{\{y(1-v)\}^{k-p-1}} \right| \cdot \right.$$

$$\left. \cdot \int_0^1 v^{p-k} (1-v)^{k-p-1} \, dv \right\}$$

$$= \mathcal{O}\left\{ \overline{\mathrm{bd}}_{a<u<w} \; \overline{\mathrm{bd}}_{0<v<1} \int_0^{w-u} \left| d_y \frac{Q(u+y, u+vy)}{\{y(1-v)\}^{k-p-1}} \right| \right\}.$$

$$= \mathcal{O}\left\{ \overline{\mathrm{bd}}_{a<u<w} \; \overline{\mathrm{bd}}_{0<v<1} \int_0^{w-u} \left| d_y \; P(y,u,v) \; S(y,u,v) \right| \right\}$$

where $S(y,u,v)$ is the product of

$$(4.3.9) \quad \{\phi(u+y)\}^{p+1} ,$$

and

$$(4.3.10) \quad (u+vy)^{-k} \{\phi(u+vy)\}^{k-p} ,$$

and $P(y,u,v) = P_{\mu,r,m}(y,u,v)$ is the product of

$$(4.3.11) \quad \left\{\frac{y \ (1-v) \ \phi'(u+vy)}{\phi(u+y) - \phi(u+vy)}\right\}^{p+1-k} ,$$

$$(4.3.12) \quad \left\{1 - \frac{\phi(u+vy)}{\phi(u+y)}\right\}^{p+1-\mu} \left\{\frac{\phi(u+vy)}{\phi(u+y)}\right\}^{\mu} ,$$

$$(4.3.13) \quad \frac{\phi^{(p+1-r-m)}(u+vy)}{\phi'(u+vy)} \left\{\frac{\phi(u+vy)}{\phi'(u+vy)}\right\}^{p-r-m} ,$$

$$(4.3.14) \quad \prod_{\nu=1}^{r} \left(\frac{\phi^{(\nu)}(u+vy)}{\phi'(u+vy)} \left\{\frac{\phi(u+vy)}{\phi'(u+vy)}\right\}^{\nu-1} \right)^{\beta_\nu} ,$$

and

$$(4.3.15) \quad \psi^{(m)}(u+vy) \left\{\frac{\phi'(u+vy)}{\phi(u+vy)}\right\}^{k-m} (u+vy)^k ,$$

where

$$0 \leq \sum_{\nu=1}^{r} \beta_\nu = \mu \leq \sum_{\nu=1}^{r} \nu\beta_\nu = r \leq p+1$$

and

$$0 \leq m \leq p+1-r .$$

Now, in view of (4.3.3), it is clear that (4.3.10) is a monotonic non-decreasing function of y in the range $[0, w-u]$ uniformly for $0 < v < 1$ and $u \geq a$.
Hence, since $\phi(t)$ is a monotonic non-decreasing function for $t \geq a$, $S(y,u,v)$ is a monotonic non-decreasing function of y in the range $[0,w-u]$ uniformly for $0 < v < 1$ and $u \geq a$, and its total variation with respect to y in the range $[0,w-u]$, is at most

$$\{\phi(w)\}^{p+1} \quad w^{-k} \{\phi(w)\}^{k-p}$$

$$= w^{-k} \{\phi(w)\}^{k+1} .$$

From condition (ii) of theorem 4.4, we can deduce that both $t\,\phi''(t) / \phi'(t)$ and $t\,\phi'(t) / \phi(t)$ are non-negative non-decreasing functions of t for $t \geq a$, and, hence, that the results of lemma 4.5 hold under the hypotheses of theorem 4.4.
Thus, $P_{\mu,r,m}(y,u,v)$ is of bounded variation with respect to y in the range $[0,w-u]$ uniformly for $0 < v < 1$ and $u \geq a$, since each of (4.3.11), (4.3.12), (4.3.13), (4.3.14) and (4.3.15) are of bounded variation with respect to y in the range $[0,w-u]$ uniformly for $0 < v < 1$ and $u \geq a$,

(4.3.11) , because of lemma 4.5 (i) and since $p+1-k > 0$,

(4.3.12) , because of lemma 4.5 (ii) and since $p+1-\mu > 0$ and $\mu \geq 0$,

(4.3.13) and (4.3.14) , because of (4.3.4) , and

(4.3.15) , because of (4.3.5) .

Hence, we can deduce that

$$q_2(w)$$

$$= \Theta\left\{ \overline{\mathrm{bd}}_{a<u<w} \ \overline{\mathrm{bd}}_{0<v<1} \int_0^{w-u} \left| d_y \, P(y,u,v) \ S(y,u,v) \right| \right\}$$

$$= \Theta\left(\overline{\mathrm{bd}}_{a<u<w} \ \overline{\mathrm{bd}}_{0<v<1} \left\{ \overline{\mathrm{bd}}_{0<y<w-u} \left| P(y,u,v) \right| . \int_0^{w-u} \left| d_y \, S(y,u,v) \right| \right. \right.$$

$$\left. \left. + \ \overline{\mathrm{bd}}_{0<y<w-u} \left| S(y,u,v) \right| . \int_0^{w-u} \left| d_y \, P(y,u,v) \right| \right\} \right)$$

$$= \Theta\left(\ . \ \{\phi(w)\}^{k+1} \, w^{-k} \right)$$

Hence, we can deduce that (4.3.8) is true.

This completes the proof of theorem 4.4 .

4.4 Proof of theorem 4.5.

The proof of theorem 4.4 can be modified to prove :-

If $\sum_{n=1}^{\infty} a_n$ is summable $[R,\lambda,k]$ and if $\psi(w)$ is a logarithmico-exponential function, tending to a non-zero finite limit as w tends to infinity,

then $\sum_{n=1}^{\infty} a_n \psi(\lambda_n)$ is summable $[R,\lambda,k]$.

Using this and results already quoted (theorem 4.1 and corollary 4.3.1) , for all $k \geq 1$, we can extend theorem 4.4 to cover logarithmico-exponential functions, $\phi(w)$, satisfying the condition

$$\frac{1}{w} \prec\!\!\!\!\prec \frac{\phi'(w)}{\phi(w)} \prec\!\!\!\!\prec 1 .$$

It therefore remains to consider the case of a logarithmico-exponential function, $\phi(w)$, satisfying the condition

$$\frac{\phi'(w)}{\phi(w)} \succ 1 .$$

For this, the argument, due to Guha [10] , is the same as that used in section 3.7 , and is not repeated.

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Appendix.

Appendix.

Given (i) $\phi(w)$ is a logarithmico-exponential function,

$$\text{(ii)} \quad \frac{1}{w} \prec \frac{\phi'(w)}{\phi(w)} \prec 1 ,$$

$$\text{(iii)} \quad \psi(w) = \left\{\frac{\phi(w)}{w\ \phi'(w)}\right\}^k ,$$

we wish to prove that

$$\text{(A)} \quad w^{-k} \{\phi(w)\}^{k-p} \succ 1 ,$$

$$\text{(B)} \quad \left\{\frac{\phi(w)}{\phi'(w)}\right\}^{n-1} \frac{\phi^{(n)}(w)}{\phi'(w)} \preccurlyeq 1 \quad \text{for} \quad n = 0, 1, \ldots , p+1 ,$$

$$\text{(C)} \quad w^k \left\{\frac{\phi'(w)}{\phi(w)}\right\}^{k-n} \psi^{(n)}(w) \prec 1$$

$$\text{for} \quad n = 0, 1, \ldots , p+1 .$$

Proofs.

(A) We have $\frac{1}{w} \prec \frac{\phi'(w)}{\phi(w)}$.

On integrating, we obtain that

$$\log w \preccurlyeq \log \phi(w) ,$$

and hence that

$$w^M \prec \phi(w) ,$$

where M is any positive number. Hence

$$w^{-M} \phi(w) \succ 1 ,$$

and choosing M to be $k/(k-p)$,

$$w^{-k} \{\phi(w)\}^{k-p} \succ 1 .$$

(B) The result is trivial in the cases $n = 0$ and $n = 1$. On differentiating

$$\frac{\phi(w)}{\phi'(w)} \prec w , \text{ (which is obtained from condition (ii))}$$

we obtain that

$$\frac{\phi(w)\ \phi''(w)}{\{\phi'(w)\}^2} \preccurlyeq 1 ,$$

i.e. $\dfrac{\phi(w)}{\phi'(w)} \ \dfrac{\phi''(w)}{\phi'(w)} \preccurlyeq 1$,

which is result (B) in the case $n = 2$.
Assume that the result holds in the case $n = N-1$. Hence

$$\phi^{(N-1)}(w) \preccurlyeq \frac{\{\phi'(w)\}^{N-1}}{\{\phi(w)\}^{N-2}} .$$

On differentiating this, we obtain that

$$\phi^{(N)}(w) \leqslant \frac{\{\phi'(w)\}^{N-2}\ \phi''(w)}{\{\phi(w)\}^{N-2}} + \frac{\{\phi'(w)\}^{N}}{\{\phi(w)\}^{N-1}}$$

$$= \frac{\{\phi'(w)\}^{N}}{\{\phi(w)\}^{N-1}} \cdot \frac{\phi(w)\ \phi''(w)}{\{\phi'(w)\}^{2}} + \frac{\{\phi'(w)\}^{N}}{\{\phi(w)\}^{N-1}}$$

$$\leqslant \frac{\{\phi'(w)\}^{N}}{\{\phi(w)\}^{N-1}}$$

since

$$\frac{\phi(w)\ \phi''(w)}{\{\phi'(w)\}^{2}} \leqslant 1 .$$

i.e. $$\left\{\frac{\phi(w)}{\phi'(w)}\right\}^{N-1} \frac{\phi^{(N)}(w)}{\phi'(w)} \leqslant 1 .$$

That is, if the result holds in the case $n = N-1$, then it holds in the case $n = N$. But the result holds in the cases $n = 0, 1, 2$. Hence it is true in the cases $n = 0, 1, \dots, p+1$.

(C) We have

$$\psi(w) = \left\{\frac{\phi(w)}{w\ \phi'(w)}\right\}^{k} .$$

Using Faa di Bruno's theorem (see, e.g., lemma 2.1), for $n = 1, 2, \dots, p+1$, $\psi^{(n)}(w)$ can be expressed as a sum of constant multiples of terms like

$$\left\{\frac{\phi(w)}{w\;\phi'(w)}\right\}^{k-\sigma} \prod_{\nu=1}^{n} \left\{\left(\frac{\partial}{\partial w}\right)^{\nu} \left(\frac{\phi(w)}{w\;\phi'(w)}\right)\right\}^{\alpha_\nu} ,$$

where $\alpha_1 , \alpha_2 , \ldots , \alpha_n$ are non-negative integers such that

$$1 \leq \sum_{\nu=1}^{n} \alpha_\nu = \sigma \leq \sum_{\nu=1}^{n} \nu\alpha_\nu = n .$$

Also, by Leibnitz's theorem on the differentiation of a product,

$$\left(\frac{\partial}{\partial w}\right)^{\nu} \left(\frac{\phi(w)}{w\;\phi'(w)}\right)$$

can be expressed as a sum of constant multiples of terms like

$$\left(\frac{\partial}{\partial w}\right)^{i} \left(\frac{1}{w}\right) \cdot \left(\frac{\partial}{\partial w}\right)^{j} \left(\frac{1}{\phi'(w)}\right) \cdot \;\phi^{(\nu-i-j)}(w) \quad ,$$

where i , j are integers such that

$$0 \leq i \leq \nu \quad ; \quad 0 \leq j \leq \nu-i \quad ,$$

which, in turn, can be expressed as a sum of constant multiples of terms like

$$N(w) = w^{-1-i}\;\phi^{(\nu-i-j)}(w) \left\{\phi'(w)\right\}^{-1-\mu} \prod_{m=1}^{j} \left\{\phi^{(m+1)}(w)\right\}^{\beta_m}$$

where $\beta_1 , \beta_2 , \ldots , \beta_j$ are non-negative integers such that

$$0 \leq \sum_{m=1}^{j} \beta_m = \mu \leq \sum_{m=1}^{j} m\beta_m = j .$$

Hence, using result (B),

$$N(w) \preccurlyeq w^{-l-i} \frac{\{\phi'(w)\}^{\nu-i-j}}{\{\phi(w)\}^{\nu-i-j-l}} \{\phi'(w)\}^{-l-\mu} \frac{\{\phi'(w)\}^{j+\mu}}{\{\phi(w)\}^{j}}$$

$$= w^{-l-i} \left\{\frac{\phi'(w)}{\phi(w)}\right\}^{\nu-i-l}$$

$$= w^{-l} \left\{\frac{\phi(w)}{w\ \phi'(w)}\right\}^{i} \left\{\frac{\phi'(w)}{\phi(w)}\right\}^{\nu-l}$$

$$\prec w^{-l} \left\{\frac{\phi'(w)}{\phi(w)}\right\}^{\nu-l}$$

using condition (ii) . Hence

$$\psi^{(n)}(w) \prec \left\{\frac{\phi(w)}{w\ \phi'(w)}\right\}^{k-\sigma} \prod_{\nu=1}^{n} \left(w^{-l} \left\{\frac{\phi'(w)}{\phi(w)}\right\}^{\nu-l}\right)^{a_\nu}$$

$$= \left\{\frac{\phi(w)}{w\ \phi'(w)}\right\}^{k-\sigma} w^{-\sigma} \left\{\frac{\phi'(w)}{\phi(w)}\right\}^{n-\sigma}$$

$$= w^{-k} \left\{\frac{\phi(w)}{\phi'(w)}\right\}^{k-n},$$

that is

$$w^{k} \left\{\frac{\phi'(w)}{\phi(w)}\right\}^{k-n} \psi^{(n)}(w) \prec 1 .$$